Chapter 6

Bases and Subbases

BASE FOR A TOPOLOGY

Let $(X, \mathcal{T})$ be a topological space. A class $\mathcal{B}$ of open subsets of X, i.e. $\mathcal{B} \subset \mathcal{T}$, is a *base* for the topology $\mathcal{T}$ iff

(i) every open set $G \in \mathcal{T}$ is the union of members of $\mathcal{B}$.

Equivalently, $\mathcal{B} \subset \mathcal{T}$ is a base for $\mathcal{T}$ iff

(ii) for any point p belonging to an open set G, there exists $B \in \mathcal{B}$ with $p \in B \subset G$.

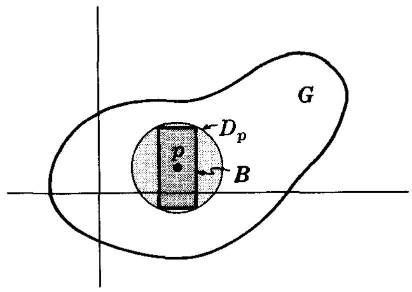
Example 1.1: The open intervals form a base for the usual topology on the line $\mathbf{R}$. For if $G \subset \mathbf{R}$ is open and $p \in G$, then by definition, $\exists$ an open interval (a, b) with $p \in (a, b) \subset G$.

Similarly, the open discs form a base for the usual topology on the plane $\mathbf{R}^2$.

Example 1.2: The open rectangles in the plane $\mathbf{R}^2$, bounded by sides parallel to the x-axis and y-axis, also form a base $\mathcal{B}$ for the usual topology on $\mathbf{R}^2$. For, let $G \subset \mathbf{R}^2$ be open and $p \in G$. Hence there exists an open disc D_p centered at p with $p \in D_p \subset G$. Then any rectangle $B \in \mathcal{B}$ whose vertices lie on the boundary of D_p satisfies

$$p \in B \subset D_p \subset G \quad \text{or} \quad p \in B \subset G$$

as indicated in the diagram. In other words, $\mathcal{B}$ satisfies (ii) above.



Example 1.3: Consider any discrete space $(X, \mathcal{D})$. Then the class $\mathcal{B} = \{\{p\} : p \in X\}$ of all singleton subsets of X is a base for the discrete topology $\mathcal{D}$ on X. For each singleton set $\{p\}$ is $\mathcal{D}$-open, since every $A \subset X$ is $\mathcal{D}$-open; furthermore, every set is the union of singleton sets. In fact any other class $\mathcal{B}^*$ of subsets of X is a base for $\mathcal{D}$ if and only if it is a superclass of $\mathcal{B}$, i.e. $\mathcal{B}^* \supset \mathcal{B}$.

We now ask the following question: Given a class $\mathcal{B}$ of subsets of a set X, when will the class $\mathcal{B}$ be a base for some topology on X? Clearly $X = \bigcup\{B : B \in \mathcal{B}\}$ is necessary since X is open in every topology on X. The next example shows that other conditions are also needed.

Example 1.4: Let $X = \{a, b, c\}$. We show that the class $\mathcal{B}$ consisting of $\{a, b\}$ and $\{b, c\}$, i.e. $\mathcal{B} = \{\{a, b\}, \{b, c\}\}$, cannot be a base for any topology on X. For then $\{a, b\}$ and $\{b, c\}$ would themselves be open and therefore their intersection $\{a, b\} \cap \{b, c\} = \{b\}$ would also be open; but $\{b\}$ is not the union of members of $\mathcal{B}$.

The next theorem gives both necessary and sufficient conditions for a class of sets to be a base for some topology.

Theorem 6.1: Let $\mathcal{B}$ be a class of subsets of a non-empty set X. Then $\mathcal{B}$ is a base for some topology on X if and only if it possesses the following two properties:

(i) $X = \bigcup\{B : B \in \mathcal{B}\}$.

(ii) For any $B, B^* \in \mathcal{B}$, $B \cap B^*$ is the union of members of $\mathcal{B}$, or, equivalently, if $p \in B \cap B^*$ then $\exists B_p \in \mathcal{B}$ such that $p \in B_p \subset B \cap B^*$.

Example 1.5: Let $\mathcal{B}$ be the class of open-closed intervals in the real line $\mathbf{R}$:

$$\mathcal{B} = \{(a, b] : a, b \in \mathbf{R},\ a < b\}$$

Clearly $\mathbf{R}$ is the union of members of $\mathcal{B}$ since every real number belongs to some open-closed intervals. In addition, the intersection $(a, b] \cap (c, d]$ of any two open-closed intervals is either empty or another open-closed interval. For example,

$$\text{if} \quad a < c < b < d \quad \text{then} \quad (a, b] \cap (c, d] = (c, b]$$

as indicated in the diagram below.

Thus the class $\mathcal{T}$ consisting of unions of open-closed intervals is a topology on $\mathbf{R}$, i.e. $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on $\mathbf{R}$. This topology $\mathcal{T}$ is called the *upper limit* topology on $\mathbf{R}$. Observe that $\mathcal{T} \neq \mathcal{U}$.

Similarly, the class of closed-open intervals,

$$\mathcal{B}^* = \{[a, b) : a, b \in \mathbf{R},\ a < b\}$$

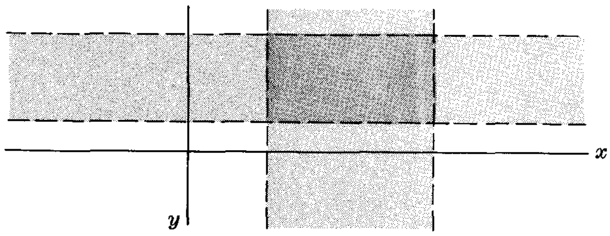
is a base for a topology $\mathcal{T}^*$ on $\mathbf{R}$ called the *lower limit* topology on $\mathbf{R}$.

SUBBASES

Let $(X, \mathcal{T})$ be a topological space. A class $\mathcal{S}$ of open subsets of X, i.e. $\mathcal{S} \subset \mathcal{T}$, is a *subbase* for the topology $\mathcal{T}$ on X iff finite intersections of members of $\mathcal{S}$ form a base for $\mathcal{T}$.

Example 2.1: Observe that every open interval (a, b) in the line $\mathbf{R}$ is the intersection of two infinite open intervals (a, ∞) and $(-\infty, b)$: $(a, b) = (a, \infty) \cap (-\infty, b)$. But the open intervals form a base for the usual topology on $\mathbf{R}$; hence the class $\mathcal{S}$ of all infinite open intervals is a subbase for $\mathbf{R}$.

Example 2.2: The intersection of a vertical and a horizontal infinite open strip in the plane $\mathbf{R}^2$ is an open rectangle as indicated in the diagram below.



But, as noted previously, the open rectangles form a base for the usual topology on $\mathbf{R}^2$. Accordingly, the class $\mathcal{S}$ of all infinite open strips is a subbase for $\mathbf{R}^2$.

TOPOLOGIES GENERATED BY CLASSES OF SETS

Let $\mathcal{A}$ be any class of subsets of a non-empty set X. As seen previously, $\mathcal{A}$ may not be a base for a topology on X. However, $\mathcal{A}$ always *generates* a topology on X in the following sense:

Theorem 6.2: Any class $\mathcal{A}$ of subsets of a non-empty set X is the subbase for a unique topology $\mathcal{T}$ on X. That is, finite intersections of members of $\mathcal{A}$ form a base for the topology $\mathcal{T}$ on X.

Example 3.1: Consider the following class of subsets of $X = \{a, b, c, d\}$:

$$\mathcal{A} = \{\{a, b\},\ \{b, c\},\ \{d\}\}$$

Finite intersections of members of $\mathcal{A}$ gives the class

$$\mathcal{B} = \{\{a, b\},\ \{b, c\},\ \{d\},\ \{b\},\ \emptyset,\ X\}$$

(Note $X \in \mathcal{B}$ since, by definition, it is the empty intersection of members of $\mathcal{A}$.) Taking unions of members of $\mathcal{B}$ gives the class

$$\mathcal{T} = \{\{a,b\}, \{b,c\}, \{d\}, \{b\}, \emptyset, X, \{a,b,d\}, \{b,c,d\}, \{b,d\}, \{a,b,c\}\}$$

which is the topology on X generated by the class $\mathcal{A}$.

Example 3.2: Let $(X, \precsim)$ be any non-empty totally ordered set. The topology on X generated by the subsets of X of the form

$$\{x \in X : x \prec p, \ p \in X\} \quad \text{or} \quad \{x \in X : p \prec x, \ p \in X\}$$

is called the *order topology* on X. Observe, by Example 2.1, that the usual topology on $\mathbf{R}$ is, in fact, identical to the (natural) order topology on $\mathbf{R}$.

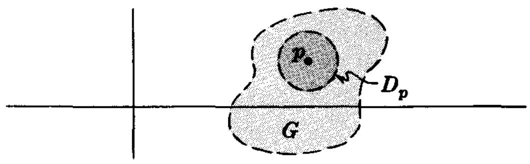
The topology generated by a class of sets can also be characterized as follows:

Proposition 6.3: Let $\mathcal{A}$ be a class of subsets of a non-empty set X. Then the topology $\mathcal{T}$ on X generated by $\mathcal{A}$ is the intersection of all topologies on X which contain $\mathcal{A}$.

LOCAL BASES

Let p be any arbitrary point in a topological space X. A class $\mathcal{B}_p$ of open sets containing p is called a *local base at p* iff for each open set G containing p, $\exists G_p \in \mathcal{B}_p$ with the property $p \in G_p \subset G$.

Example 4.1: Consider the usual topology on the plane $\mathbf{R}^2$ and any point $p \in \mathbf{R}^2$. Then the class $\mathcal{B}_p$ of all open discs centered at p is a local base at p. For, as proven previously, any open set G containing p also contains an open disc D_p whose center is p.



Similarly, the class of all open intervals $(a - \delta, a + \delta)$ in the line $\mathbf{R}$ with center $a \in \mathbf{R}$ is a local base at the point a.

The following relationship between a base ("in the large") for a topology and a local base ("in the small") at a point clearly holds:

Proposition 6.4: Let $\mathcal{B}$ be a base for a topology $\mathcal{T}$ on X and let $p \in X$. Then the members of the base $\mathcal{B}$ which contain p form a local base at the point p.

Some concepts previously defined in terms of the open sets containing a point p can also be defined merely in terms of the members of a local base at p. For example,

Proposition 6.5: A point p in a topological space X is an accumulation point of $A \subset X$ iff each member of some local base $\mathcal{B}_p$ at p contains a point of A different from p.

Proposition 6.6: A sequence $\langle a_1, a_2, \ldots \rangle$ of points in a topological space X converges to $p \in X$ iff each member of some local base $\mathcal{B}_p$ at p contains almost all of the terms of the sequence.

The previous three propositions imply the following useful corollary.

Corollary 6.7: Let $\mathcal{B}$ be a base for a topology $\mathcal{T}$ on X. Then:

(i) $p \in X$ is an accumulation point of $A \subset X$ iff each open base set $B \in \mathcal{B}$ containing p contains a point of A different from p;

(ii) a sequence $\langle a_1, a_2, \ldots \rangle$ of points in X converges to $p \in X$ iff each open base set $B \in \mathcal{B}$ containing p contains almost all of the terms of the sequence.

Example 4.2: Consider the lower limit topology $\mathcal{T}$ on the real line $\mathbf{R}$ which has as a base the class of closed-open intervals $[a, b)$, and let $A = (0,1)$. Note that $G = [1,2)$ is a $\mathcal{T}$-open set containing $1 \in \mathbf{R}$ for which $G \cap A = \emptyset$; hence 1 is not a limit point of A. On the other hand, $0 \in \mathbf{R}$ is a limit point of A since any open base set $[a, b)$ containing 0, i.e. for which $a \leq 0 < b$, contains points of A other than 0.

Solved Problems

BASES

1. Show the equivalence of both definitions of a base for a topology, that is, if $\mathcal{B}$ is a subclass of $\mathcal{T}$ then the following statements are equivalent:

   (i) Each $G \in \mathcal{T}$ is the union of members of $\mathcal{B}$.

   (ii) For any point p belonging to an open set G, $\exists B_p \in \mathcal{B}$ such that $p \in B_p \subset G$.

Solution:

If $G = \cup_i B_i$ where $B_i \in \mathcal{B}$, then each point $p \in G = \cup_i B_i$ belongs to at least one member B_{i_0} in the union; so

$$p \in B_{i_0} \subset \cup_i B_i = G$$

On the other hand, if for each $p \in G$, $\exists B_p \in \mathcal{B}$ such that $p \in B_p \subset G$, then

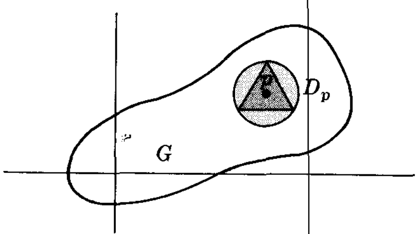
$$G = \bigcup\{B_p : p \in G\}$$

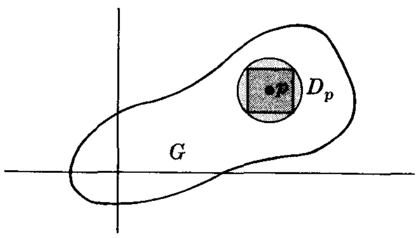
and G is the union of members of $\mathcal{B}$.

2. Determine whether or not each of the following classes of subsets of the plane $\mathbf{R}^2$ is a base for the usual topology on $\mathbf{R}^2$: (i) the class of open equilateral triangles; (ii) the class of open squares with horizontal and vertical sides.

Solution:

Both of the above classes are a base for the usual topology on $\mathbf{R}^2$. For let G be an open subset of $\mathbf{R}^2$ and let $p \in G$. Then $\exists$ an open disc D_p centered at p such that $p \in D_p \subset G$. Observe that either an equilateral triangle or a square can be inscribed in D_p as indicated in the diagrams below.





Hence each class satisfies the second definition of a base for a topology.

3. Let $\mathcal{B}$ be a base for a topology $\mathcal{T}$ on X and let $\mathcal{B}^*$ be a class of open sets containing $\mathcal{B}$, i.e. $\mathcal{B} \subset \mathcal{B}^* \subset \mathcal{T}$. Show that $\mathcal{B}^*$ is also a base for $\mathcal{T}$.

Solution:

Let G be an open subset of X. Since $\mathcal{B}$ is a base for $(X, \mathcal{T})$, G is the union of members of $\mathcal{B}$, i.e. $G = \cup_i B_i$ where $B_i \in \mathcal{B}$. But $\mathcal{B} \subset \mathcal{B}^*$; hence each $B_i \in \mathcal{B}$ also belongs to $\mathcal{B}^*$. So G is the union of members of $\mathcal{B}^*$, and therefore $\mathcal{B}^*$ is also a base for $(X, \mathcal{T})$.

4. Let X be a discrete space and let $\mathcal{B}$ be the class of all singleton subsets of X, i.e. $\mathcal{B} = \{\{p\} : p \in X\}$. Show that any class $\mathcal{B}^*$ of subsets of X is a base for X if and only if it is a superclass of $\mathcal{B}$.

Solution:

Suppose $\mathcal{B}^*$ is a base for X. Since any singleton set $\{p\}$ is open in a discrete space, $\{p\}$ must be a union of members of $\mathcal{B}^*$. But a singleton set can only be the union of itself or itself with the empty set $\emptyset$. Hence $\{p\}$ must be a member of $\mathcal{B}^*$, so $\mathcal{B} \subset \mathcal{B}^*$.

On the other hand, since $\mathcal{B}$ is a base for the discrete space X (see Example 1.3), any superset of $\mathcal{B}$ is also a base for X.

5. Prove Theorem 6.1: Let $\mathcal{B}$ be a class of subsets of a non-empty set X. Then $\mathcal{B}$ is a base for some topology on X if and only if it satisfies the following two properties:

(i) $X = \bigcup\{B : B \in \mathcal{B}\}$.

(ii) For any $B, B^* \in \mathcal{B}$, $B \cap B^*$ is the union of members of $\mathcal{B}$, or, equivalently, if $p \in B \cap B^*$ then $\exists B_p \in \mathcal{B}$ such that $p \in B_p \subset B \cap B^*$.

Solution:

Suppose $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on X. Since X is open, X is the union of members of $\mathcal{B}$. Hence X is the union of all the members of $\mathcal{B}$, i.e. $X = \bigcup\{B : B \in \mathcal{B}\}$. Furthermore, if $B, B^* \in \mathcal{B}$ then, in particular, B and B^* are open. Hence the intersection $B \cap B^*$ is also open and, since $\mathcal{B}$ is a base for $\mathcal{T}$, it is the union of members of $\mathcal{B}$. Thus (i) and (ii) are satisfied.

Conversely, suppose $\mathcal{B}$ is a class of subsets of X which satisfy (i) and (ii) above. Let $\mathcal{T}$ be the class of all subsets of X which are unions of members of $\mathcal{B}$. We claim that $\mathcal{T}$ is a topology on X. Observe that $\mathcal{B} \subset \mathcal{T}$ will be a base for this topology.

By (i), $X = \bigcup\{B : B \in \mathcal{B}\}$; so $X \in \mathcal{T}$. Note that $\emptyset$ is the union of the empty subclass of $\mathcal{B}$, i.e. $\emptyset = \bigcup\{B : B \in \emptyset \subset \mathcal{B}\}$; hence $\emptyset \in \mathcal{T}$, and so $\mathcal{T}$ satisfies $[\mathbf{O_1}]$.

Now let $\{G_i\}$ be a class of members of $\mathcal{T}$. By definition of $\mathcal{T}$, each G_i is the union of members of $\mathcal{B}$; hence the union $\cup_i G_i$ is also the union of members of $\mathcal{B}$ and so belongs to $\mathcal{T}$. Thus $\mathcal{T}$ satisfies $[\mathbf{O_2}]$.

Lastly, suppose $G, H \in \mathcal{T}$. We need to show that $G \cap H$ also belongs to $\mathcal{T}$. By definition of $\mathcal{T}$, there exist two subclasses $\{B_i : i \in I\}$ and $\{B_j : j \in J\}$ of $\mathcal{B}$ such that $G = \cup_i B_i$ and $H = \cup_j B_j$. Then, by the distributive laws,

$$G \cap H = (\cup_i B_i) \cap (\cup_j B_j) = \bigcup\{B_i \cap B_j : i \in I,\ j \in J\}$$

But by (ii), $B_i \cap B_j$ is the union of members of $\mathcal{B}$; hence $G \cap H = \bigcup\{B_i \cap B_j : i \in I, j \in J\}$ is also the union of members of $\mathcal{B}$ and so belongs to $\mathcal{T}$ which therefore satisfies $[\mathbf{O_3}]$. Hence $\mathcal{T}$ is a topology on X with base $\mathcal{B}$.

6. Let $\mathcal{B}$ and $\mathcal{B}^*$ be bases, respectively, for topologies $\mathcal{T}$ and $\mathcal{T}^*$ on a set X. Suppose that each $B \in \mathcal{T}$ is the union of members of $\mathcal{B}^*$. Show that $\mathcal{T}$ is coarser than $\mathcal{T}^*$, i.e. $\mathcal{T} \subset \mathcal{T}^*$.

Solution:

Let G be a $\mathcal{T}$-open set. Then G is the union of members of $\mathcal{B}$, i.e. $G = \cup_i B_i$ where $B_i \in \mathcal{T}$. But, by hypothesis, each $B_i \in \mathcal{B}$ is the union of members of $\mathcal{B}^*$, and so $G = \cup_i B_i$ is also the union of members of $\mathcal{B}^*$ which are $\mathcal{T}^*$-open sets. Hence G is also a $\mathcal{T}^*$-open set, and so $\mathcal{T} \subset \mathcal{T}^*$.

7. Show that the usual topology $\mathcal{U}$ on the real line $\mathbf{R}$ is coarser than the upper limit topology $\mathcal{T}$ on $\mathbf{R}$ which has as a base the class of open-closed intervals $(a, b]$.

Solution:

Note first that any open interval is the union of open-closed intervals. For example,

$$(a, b) = \bigcup\{(a, b - 1/n] : n \in \mathbf{N}\}$$

Since the class of open intervals is a base for $\mathcal{U}$, by the preceding problem, $\mathcal{U} \subset \mathcal{T}$, i.e. any $\mathcal{U}$-open set is also $\mathcal{T}$-open.

8. Consider the upper limit topology $\mathcal{T}$ on the real line $\mathbf{R}$ which has as a base the class of open-closed intervals $(a, b]$. (i) Show that the open infinite interval $(4, \infty)$ and the closed infinite interval $(-\infty, 2]$ are $\mathcal{T}$-open sets. (ii) Show that any open infinite interval (a, ∞) and any closed infinite interval $(-\infty, b]$ are $\mathcal{T}$-open sets. (iii) Show that any open-closed interval $(a, b]$ is both $\mathcal{T}$-open and $\mathcal{T}$-closed.

Solution:

(i) Observe that
$$(4, \infty) = (4,5] \cup (4,6] \cup (4,7] \cup (4,8] \cup \cdots$$
$$(-\infty, 2] = (0,2] \cup (-1,2] \cup (-2,2] \cup \cdots$$

Hence each is $\mathcal{T}$-open since each is the union of members of the base for $\mathcal{T}$.

(ii) Similarly,
$$(a, \infty) = (a, a+1] \cup (a, a+2] \cup (a, a+3] \cup \cdots$$
$$(-\infty, b] = (b-1, b] \cup (b-2, b] \cup (b-3, b] \cup (b-4, b] \cup \cdots$$

Hence each is $\mathcal{T}$-open.

(iii) $(a, b]^c = (-\infty, a] \cup (b, \infty)$, and the two intervals on the right are open, so their union is open and therefore $(a, b]$ is closed. But $(a, b]$ belongs to the base for $\mathcal{T}$ and so is also open.

SUBBASES, TOPOLOGIES GENERATED BY CLASSES OF SETS

9. Let $X = \{a, b, c, d, e\}$ and let $\mathcal{A} = \{\{a, b, c\}, \{c, d\}, \{d, e\}\}$. Find the topology on X generated by $\mathcal{A}$.

Solution:

First compute the class $\mathcal{B}$ of all finite intersections of sets in $\mathcal{A}$:
$$\mathcal{B} = \{X, \{a,b,c\}, \{c,d\}, \{d,e\}, \{c\}, \{d\}, \emptyset\}$$

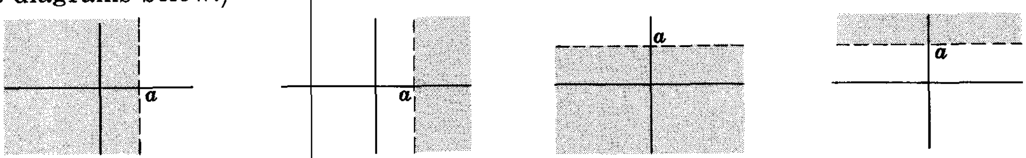
(Note that $X \in \mathcal{B}$, since by definition X is the empty intersection of members of $\mathcal{A}$.) Taking unions of members of $\mathcal{B}$ gives the class
$$\mathcal{T} = \{X, \{a,b,c\}, \{c,d\}, \{d,e\}, \{c\}, \{d\}, \emptyset, \{a,b,c,d\}, \{c,d,e\}\}$$
which is the topology on X generated by $\mathcal{A}$.

10. Determine the topology $\mathcal{T}$ on the real line $\mathbf{R}$ generated by the class $\mathcal{A}$ of all closed intervals $[a, a+1]$ with length one.

Solution:

Let p be any point in $\mathbf{R}$. Note that the closed intervals $[p-1, p]$ and $[p, p+1]$ belong to $\mathcal{A}$ as they have length one. Hence
$$[p-1, p] \cap [p, p+1] = \{p\}$$
belongs to the topology $\mathcal{T}$, i.e. all singleton sets $\{p\}$ are $\mathcal{T}$-open, and so $\mathcal{T}$ is the discrete topology on X.

11. Let $\mathcal{A}$ be the class of all open half-planes H in the plane $\mathbf{R}^2$ of the form
$$H = \{\langle x, y\rangle : x < a, \text{ or } x > a, \text{ or } y < a, \text{ or } y > a\}$$
(See diagrams below.)



$x < a$ $x > a$ $y < a$ $y > a$

Find the topology on $\mathbf{R}^2$ generated by $\mathcal{A}$.

Solution:

Observe that every open rectangle $B = \{\langle x,y\rangle : a < x < b,\ c < y < d\}$ is the intersection of the four half-planes

$$H_1 = \{\langle x,y\rangle : a < x\} \qquad H_3 = \{\langle x,y\rangle : c < y\}$$
$$H_2 = \{\langle x,y\rangle : x < b\} \qquad H_4 = \{\langle x,y\rangle : y < d\}$$

Since each $H \in \mathcal{A}$ is $\mathcal{U}$-open, and since the class of all open rectangles B is a base for the usual topology $\mathcal{U}$ on $\mathbf{R}^2$, the class $\mathcal{A}$ is a subbase for $\mathcal{U}$. That is, $\mathcal{A}$ generates the usual topology on the plane $\mathbf{R}^2$.

12. Consider the discrete topology $\mathcal{D}$ on $X = \{a,b,c,d,e\}$. Find a subbase $\mathcal{S}$ for $\mathcal{D}$ which does not contain any singleton sets.

Solution:

Recall that any class $\mathcal{B}$ of subsets of X is a base for the discrete topology $\mathcal{D}$ on X iff it contains all singleton subsets of X. Hence $\mathcal{S}$ is a subbase for $\mathcal{D}$ iff finite intersections of members of $\mathcal{S}$ gives $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$ and $\{e\}$. So $\mathcal{S} = \{\{a,b\}, \{b,c\}, \{c,d\}, \{d,e\}, \{e,a\}\}$ is a subbase for $\mathcal{D}$.

13. Let $\mathcal{S}$ be a subbase for a topology $\mathcal{T}$ on X and let A be a subset of X. Show that the class $\mathcal{S}_A = \{A \cap S : S \in \mathcal{S}\}$ is a subbase for the relative topology $\mathcal{T}_A$ on A.

Solution:

Let H be a $\mathcal{T}_A$-open subset of A. Then $H = A \cap G$ where G is a $\mathcal{T}$-open subset of X. By hypothesis, $\mathcal{S}$ is a subbase for $\mathcal{T}$; so

$$G = \cup_i (S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_{n_i}}) \qquad \text{where } S_{i_k} \in \mathcal{S}$$

Hence
$$H = A \cap G = A \cap [\cup_i (S_{i_1} \cap \cdots \cap S_{i_{n_i}})]$$
$$= \cup_i [(A \cap S_{i_1}) \cap \cdots \cap (A \cap S_{i_{n_i}})]$$

Thus H is the union of finite intersections of members of $\mathcal{S}_A$ and therefore $\mathcal{S}_A$ is a subbase for $\mathcal{T}_A$.

14. Show that all intervals $(a,1]$ and $[0,b)$, where $0 < a, b < 1$, form a subbase for the relative usual topology on the unit interval $I = [0,1]$.

Solution:

Recall that the infinite open intervals (a,∞) and $(-\infty,b)$ form a subbase for the usual topology on the real line $\mathbf{R}$. The intersection of these infinite open intervals with $I = [0,1]$ are the sets $\emptyset$, I, $(a,1]$ and $[0,b)$ which, by the preceding problem, form a subbase for $I = [0,1]$. But we can exclude the empty set $\emptyset$ and the whole space I from any subbase; so the intervals $(a,1]$ and $[0,b)$ form a subbase for I.

15. Show that if $\mathcal{S}$ is a subbase for topologies $\mathcal{T}$ and $\mathcal{T}^*$ on X, then $\mathcal{T} = \mathcal{T}^*$.

Solution:

Suppose $G \in \mathcal{T}$. Since $\mathcal{S}$ is a subbase for $\mathcal{T}$, $G = \cup_i (S_{i_1} \cap \cdots \cap S_{i_{n_i}})$ where $S_{i_k} \in \mathcal{S}$.

But $\mathcal{S}$ is also a subbase for $\mathcal{T}^*$ and so $\mathcal{S} \subset \mathcal{T}^*$; hence each $S_{i_k} \in \mathcal{T}^*$. Since $\mathcal{T}^*$ is a topology, $S_{i_1} \cap \cdots \cap S_{i_{n_i}} \in \mathcal{T}^*$ and hence $G \in \mathcal{T}^*$. Thus $\mathcal{T} \subset \mathcal{T}^*$. Similarly $\mathcal{T}^* \subset \mathcal{T}$, and so $\mathcal{T} = \mathcal{T}^*$.

16. Prove Theorem 6.2: Any class $\mathcal{A}$ of subsets of a non-empty set X is the subbase for a unique topology on X. That is, finite intersections of members of $\mathcal{A}$ form a base for a topology $\mathcal{T}$ on X.

Solution:

We show that the class $\mathcal{B}$ of finite intersections of members of $\mathcal{A}$ satisfies the two conditions in Theorem 6.1 for it to be a base for a topology on X:

(i) $X = \bigcup\{B : B \in \mathcal{B}\}$.

(ii) For any $G, H \in \mathcal{B}$, $G \cap H$ is the union of members of $\mathcal{B}$.

Note $X \in \mathcal{B}$, since X by definition is the empty intersection of members of $\mathcal{A}$; so

$$X = \bigcup\{B : B \in \mathcal{B}\}$$

Furthermore, if $G, H \in \mathcal{B}$, then G and H are finite intersections of members of $\mathcal{A}$. Hence $G \cap H$ is also a finite intersection of members of $\mathcal{A}$ and therefore belongs to $\mathcal{B}$. Accordingly, $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on X for which $\mathcal{A}$ is a subbase. The preceding problem shows that $\mathcal{T}$ is unique.

17. Prove Proposition 6.3: Let $\mathcal{A}$ be a class of subsets of a non-empty set X. Then the topology $\mathcal{T}$ on X generated by $\mathcal{A}$ is the intersection of all topologies on X which contain $\mathcal{A}$.

Solution:

Let $\{\mathcal{T}_i\}$ be the collection of topologies on X containing $\mathcal{A}$, and let $\mathcal{T}^* = \cap_i \mathcal{T}_i$. Note that $\mathcal{A} \subset \mathcal{T}^*$. We want to prove that $\mathcal{T} = \mathcal{T}^*$. Since $\mathcal{T}$ is a topology containing $\mathcal{A}$, and $\mathcal{T}^*$ is the intersection of all such topologies, we have $\mathcal{T}^* \subset \mathcal{T}$.

On the other hand, suppose $G \in \mathcal{T}$. Then by the definition of $\mathcal{T}$,

$$G = \cup_i (S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i n_i}) \qquad \text{where} \qquad S_{i_k} \in \mathcal{A}$$

But $\mathcal{A} \subset \mathcal{T}^*$, hence each $S_{i_k} \in \mathcal{T}^*$. Accordingly, $S_{i_1} \cap \cdots \cap S_{i n_i} \in \mathcal{T}^*$ and so

$$G = \cup_i (S_{i_1} \cap \cdots \cap S_{i n_i}) \in \mathcal{T}^*$$

We have shown that $G \in \mathcal{T}$ implies $G \in \mathcal{T}^*$; hence $\mathcal{T} \subset \mathcal{T}^*$. Consequently $\mathcal{T} = \mathcal{T}^*$.

LOCAL BASES

18. Prove Proposition 6.5: A point p in a topological space $(X, \mathcal{T})$ is an accumulation point of $A \subset X$ iff each member of some local base $\mathcal{B}_p$ at p contains a point of A different from p.

Solution:

Recall $p \in X$ is an accumulation point of A iff $(G \setminus \{p\}) \cap A \neq \emptyset$ for all $G \in \mathcal{T}$ such that $p \in G$. But $\mathcal{B}_p \subset \mathcal{T}$, so in particular $(B \setminus \{p\}) \cap A \neq \emptyset$ for all $B \in \mathcal{B}_p$.

Conversely, suppose $(B \setminus \{p\}) \cap A \neq \emptyset$ for all $B \in \mathcal{B}_p$, and let G be any open subset of X containing p. Then $\exists B_0 \in \mathcal{B}_p$ for which $p \in B_0 \subset G$. But then

$$(G \setminus \{p\}) \cap A \supset (B_0 \setminus \{p\}) \cap A \neq \emptyset$$

So $(G \setminus \{p\}) \cap A \neq \emptyset$, or p is an accumulation point of A.

19. Prove Proposition 6.6: A sequence $\langle a_1, a_2, \ldots \rangle$ of points in a topological space $(X, \mathcal{T})$ converges to $p \in X$ iff each member of some local base $\mathcal{B}_p$ at p contains almost all of the terms of the sequence.

Solution:

Recall that $a_n \to p$ iff every open set $G \in \mathcal{T}$ containing p contains almost all the terms of the sequence. But $\mathcal{B}_p \subset \mathcal{T}$, so in particular each $B \in \mathcal{B}_p$ contains almost all the terms of the sequence.

On the other hand, suppose every $B \in \mathcal{B}_p$ contains almost all the terms of the sequence, and let G be any open set containing p. Then $\exists B_0 \in \mathcal{B}_p$ for which $p \in B_0 \subset G$. Hence G also contains almost all the terms of the sequence, and so $\langle a_n \rangle$ converges to p.

20. Show that every point p in a discrete space X has a finite local base.

Solution:

Note that the singleton set $\{p\}$ is open since every subset of a discrete space is open. Accordingly the class $\mathcal{B}_p = \{\{p\}\}$, i.e. the class consisting of the singleton set $\{p\}$, is a local base at p since every open set G containing p must be a superset of $\{p\}$.

21. Consider the upper limit topology $\mathcal{T}$ on the real line $\mathbf{R}$ which has as a base the class of open-closed intervals $(a, b]$. Determine whether or not each of the following sequences converges to 0:

(i) $\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle$ (ii) $\langle -1, -\frac{1}{2}, -\frac{1}{3}, \ldots \rangle$

Solution:

(i) No. For the $\mathcal{T}$-open set $(-2, 0]$ containing 0 does not contain any term of the sequence.

(ii) Yes. For any open basic set $(a, b]$ containing 0, i.e. for which $a < 0 \leq b$, $\exists n_0 \in N$ such that $a < -1/n_0 < 0$. Hence $n > n_0$ implies $-1/n \in (a, b]$.

Supplementary Problems

BASES FOR TOPOLOGIES

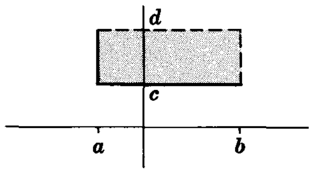
22. Show that the class of closed intervals $[a, b]$, where a and b are rational and $a < b$, is not a base for a topology on the real line $\mathbf{R}$.

23. Show that the class of closed intervals $[a, b]$, where a is rational and b is irrational and $a < b$, is a base for a topology on the real line $\mathbf{R}$.

24. Let $\mathcal{B}$ be a base for a topology $\mathcal{T}$ on X and let $A \subset X$. Show that the class $\mathcal{B}_A = \{A \cap G : G \in \mathcal{B}\}$ is a base for the relative topology $\mathcal{T}_A$ on A.

25. Let $\mathcal{B}$ be the class of half-open rectangles in the plane $\mathbf{R}^2$ indicated in the diagram on the right, i.e. of the form

$$\{\langle x, y\rangle : a \leq x < b,\ c \leq y < d\}$$

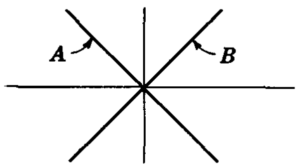


(i) Show that $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on $\mathbf{R}^2$.

(ii) Show that the relative topology $\mathcal{T}_A$ on the line

$$A = \{\langle x, y\rangle : x + y = 0\}$$

is the discrete topology on A.



(iii) Show that the relative topology $\mathcal{T}_B$ on the line

$$B = \{\langle x, y\rangle : x = y\}$$

is not the discrete topology on B.

26. Let $\mathcal{B}$ be a class of subsets of a non-empty set X totally ordered by set inclusion. Show that $\mathcal{B}$ is a base for a topology on X provided that $X = \bigcup\{B : B \in \mathcal{B}\}$.

27. Show that a topology $\mathcal{T}$ on X is finite if and only if $\mathcal{T}$ has a finite base.

SUBBASES

28. Let $X = \{a, b, c, d, e\}$. Find the topology $\mathcal{T}$ on X generated by $\mathcal{A} = \{\{a\}, \{a, b, c\}, \{c, d\}\}$.

29. Determine the smallest subbase $\mathcal{S}$ for the cofinite topology $\mathcal{T}$ on any non-empty set X.

30. Let $\mathcal{S}$ be the class of all closed intervals $[a, b]$ where a and b are rational, i.e. $a, b \in \mathbf{Q}$, and $a < b$. Show that $\mathcal{S} \cup \{\{p\} : p \in \mathbf{Q}\}$ is a base for the topology $\mathcal{T}$ on the real line $\mathbf{R}$ generated by $\mathcal{S}$.

31. Show that if $\mathcal{S}$ is a subbase for a topology $\mathcal{T}$ on X, then $\mathcal{S} \setminus \{X, \emptyset\}$ is also a subbase for $\mathcal{T}$.

32. Let $\mathcal{T}$ and $\mathcal{T}^*$ be the topologies on X generated respectively by $\mathcal{A}$ and $\mathcal{A}^*$.
Show that: (i) $\mathcal{A} \subset \mathcal{A}^*$ implies $\mathcal{T} \subset \mathcal{T}^*$; and (ii) $\mathcal{A} \subset \mathcal{A}^* \subset \mathcal{T}$ implies $\mathcal{T} = \mathcal{T}^*$.

33. Let $\mathcal{S}$ be a subbase for a topological space X and let $G \subset X$ be an open set containing a point $p \in X$. Show that there exists a finite number of members of $\mathcal{S}$, say $S_1, S_2, \ldots, S_m$, with the property that $p \in S_1 \cap S_2 \cap \cdots \cap S_m \subset G$.

LOCAL BASES

34. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{A}$ be a $\mathcal{T}$-local base at $p \in X$. Consider any subset A of X such that $p \in A \subset X$, and consider the relative topology $\mathcal{T}_A$ on A. Show that the following class of subsets of A is a $\mathcal{T}_A$-local base at $p \in A$: $\mathcal{A}_A = \{A \cap G : G \in \mathcal{A}\}$.

35. Let X be a topological space, let $p \in X$, let $\mathcal{N}_p$ be the neighborhood system at p and let $\mathcal{B}_p$ be a local base at p. Show that every neighborhood of p contains a member of the local base at p; i.e. for every $N \in \mathcal{N}_p$, $\exists G \in \mathcal{B}_p$ for which $G \subset N$.

36. Show that if a point p has a finite local base $\mathcal{B}_p$ then it also has a local base consisting of exactly one set.

37. Consider the upper limit topology $\mathcal{T}$ on the real line $\mathbf{R}$ which has as a base the class of open-closed intervals $(a, b]$. Determine whether or not each of the following sequences converges:
(i) $\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle$, (ii) $\langle -1, -\frac{1}{2}, -\frac{1}{3}, \ldots \rangle$, (iii) $\langle -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots \rangle$.

38. Let $\mathcal{T}$ be the topology on the real line $\mathbf{R}$ generated by the class $\mathcal{S}$ of all closed intervals $[a, b]$ where a and b are rational (see Problem 30).

(i) Determine whether or not each of the following sequences converges:
(*a*) $\langle 2+\frac{1}{2}, 2+\frac{1}{3}, 2+\frac{1}{4}, \ldots \rangle$, (*b*) $\langle \sqrt{2}+\frac{1}{2}, \sqrt{2}+\frac{1}{3}, \sqrt{2}+\frac{1}{4}, \ldots \rangle$.

(ii) Determine the closure of each of the following subsets of $\mathbf{R}$:
(*a*) $(2, 4)$, (*b*) $(\sqrt{2}, 5]$, (*c*) $(-3, \pi)$, (*d*) $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.

(iii) Show that any finite subset of $\mathbf{R}$ is $\mathcal{T}$-closed.

39. Let $\mathcal{S}$ be a subbase for a topological space X and let $p \in X$.

(i) Show by a counterexample that the class $\mathcal{S}_p = \{S \in \mathcal{S} : p \in S\}$ need not be a local base at p.

(ii) Show that finite intersections of members of $\mathcal{S}_p$ do form a local base at p.

(iii) Show that a sequence $\langle a_n \rangle$ in X converges to p if and only if every $S \in \mathcal{S}_p$ contains all except a finite number of the terms of the sequence.

Answers to Supplementary Problems

28. $\mathcal{T} = \{X, \emptyset, \{a\}, \{c\}, \{a,c\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{a,b,c,d\}\}$

37. (i) No (ii) Yes (iii) No.

38. (i): (*a*) No, (*b*) Yes. (ii): (*a*) $(2,4)$, (*b*) $[\sqrt{2}, 5]$, (*c*) $(-3, \pi]$, (*d*) A.

39. (ii) *Hint.* Use Problem 33.

Chapter 7

Continuity and Topological Equivalence

CONTINUOUS FUNCTIONS

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}^*)$ be topological spaces. A function f from X into Y is *continuous relative to* $\mathcal{T}$ and $\mathcal{T}^*$, or $\mathcal{T}$-$\mathcal{T}^*$ *continuous*, or simply *continuous*, iff the inverse image $f^{-1}[H]$ of every $\mathcal{T}^*$-open subset H of Y is a $\mathcal{T}$-open subset of X, that is, iff

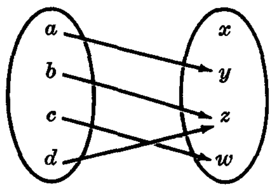
$$H \in \mathcal{T}^* \quad \text{implies} \quad f^{-1}[H] \in \mathcal{T}$$

We shall write $f : (X, \mathcal{T}) \to (Y, \mathcal{T}^*)$ for a function from X into Y when it is convenient to indicate the topologies involved.

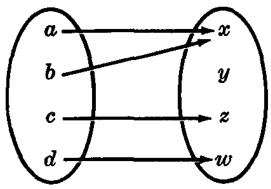
Example 1.1: Consider the following topologies on $X = \{a, b, c, d\}$ and $Y = \{x, y, z, w\}$ respectively:

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}, \quad \mathcal{T}^* = \{Y, \emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z, w\}\}$$

Also consider the functions $f : X \to Y$ and $g : X \to Y$ defined by the diagrams below:



f



g

The function f is continuous since the inverse of each member of the topology $\mathcal{T}^*$ on Y is a member of the topology $\mathcal{T}$ on X. The function g is not continuous since $\{y, z, w\} \in \mathcal{T}^*$, i.e. is an open subset of Y, but its inverse image $g^{-1}[\{y, z, w\}] = \{c, d\}$ is not an open subset of X, i.e. does not belong to $\mathcal{T}$.

Example 1.2: Consider any discrete space $(X, \mathcal{D})$ and any topological space $(Y, \mathcal{T})$. Then every function $f : X \to Y$ is $\mathcal{D}$-$\mathcal{T}$ continuous. For if H is any open subset of Y, its inverse $f^{-1}[H]$ is an open subset of X since every subset of a discrete space is open.

Example 1.3: Let $f : X \to Y$ where X and Y are topological spaces, and let $\mathcal{B}$ be a base for the topology on Y. Suppose for each member $B \in \mathcal{B}$, $f^{-1}[B]$ is an open subset of X; then f is a continuous function. For let H be an open subset of Y; then $H = \cup_i B_i$, a union of members of $\mathcal{B}$. But

$$f^{-1}[H] = f^{-1}[\cup_i B_i] = \cup_i f^{-1}[B_i]$$

and each $f^{-1}[B_i]$ is open by hypothesis; hence $f^{-1}[H]$ is the union of open sets and is therefore open. Accordingly, f is continuous.

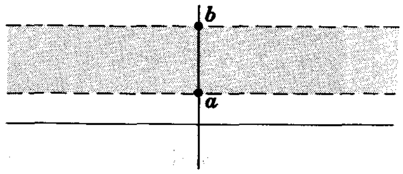
We formally state the result of the preceding example.

Proposition 7.1: A function $f : X \to Y$ is continuous iff the inverse of each member of a base $\mathcal{B}$ for Y is an open subset of X.

This proposition can in fact be strengthened as follows:

Theorem 7.2: Let $\mathcal{S}$ be a subbase for a topological space Y. Then a function $f : X \to Y$ is continuous iff the inverse of each member of $\mathcal{S}$ is an open subset of X.

Example 1.4: The projection mappings from the plane $\mathbf{R}^2$ into the line $\mathbf{R}$ are both continuous relative to the usual topologies. Consider, for example, the projection $\pi : \mathbf{R}^2 \to \mathbf{R}$ defined by $\pi(\langle x, y \rangle) = y$. Then the inverse of any open interval (a, b) is an infinite open strip as illustrated below:



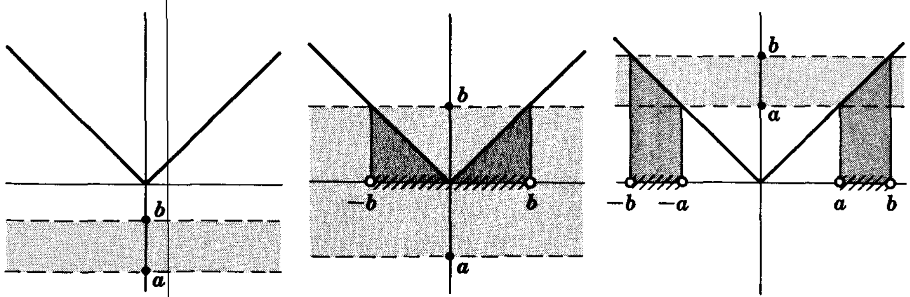
$\pi^{-1}[(a, b)]$ is shaded

Hence by Proposition 7.1, the inverse of every open subset of $\mathbf{R}$ is open in $\mathbf{R}^2$, i.e. π is continuous.

Example 1.5: The absolute value function f on $\mathbf{R}$, i.e. $f(x) = |x|$ for $x \in \mathbf{R}$, is continuous. For if $A = (a, b)$ is an open interval in $\mathbf{R}$, then

$$f^{-1}[A] = \begin{cases} \emptyset & \text{if } a < b \leq 0 \\ (-b, b) & \text{if } a < 0 < b \\ (-b, -a) \cup (a, b) & \text{if } 0 \leq a < b \end{cases}$$

as illustrated below. In each case $f^{-1}[A]$ is open; hence f is continuous.



$f^{-1}[A] = \emptyset$ $\quad$ $f^{-1}[A] = (-b, b)$ $\quad$ $f^{-1}[A] = (-b, -a) \cup (a, b)$

Continuous functions can be characterized by their behavior with respect to closed sets, as follows:

Theorem 7.3: A function $f : X \to Y$ is continuous if and only if the inverse image of every closed subset of Y is a closed subset of X.

CONTINUOUS FUNCTIONS AND ARBITRARY CLOSENESS

Let X be a topological space. A point $p \in X$ is said to be *arbitrarily close* to a set $A \subset X$ if

either (i) $p \in A$ or (ii) p is an accumulation point of A

Recall that $\bar{A} = A \cup A'$; so the closure of A consists precisely of those points in X which are arbitrarily close to A. Recall also that $\bar{A} = A^\circ \cup \mathrm{b}(A)$; hence p is arbitrarily close to A if p is either an interior or a boundary point of A.

Continuous functions can also be characterized as those functions which *preserve arbitrary closeness*, namely,

Theorem 7.4: A function $f : X \to Y$ is continuous if and only if, for any $p \in X$ and any $A \subset X$,

$$p \text{ arbitrarily close to } A \;\Rightarrow\; f(p) \text{ arbitrarily close to } f[A]$$

or

$$p \in \bar{A} \;\Rightarrow\; f(p) \in \overline{f[A]}$$

or

$$f[\bar{A}] \subset \overline{f[A]}$$

CONTINUITY AT A POINT

Continuity as we have defined it is a *global* property, that is, it restricts the way in which a function behaves on the entire set X. There also exists a corresponding local concept of *continuity at a point.*

A function $f: X \to Y$ is continuous at $p \in X$ iff the inverse image $f^{-1}[H]$ of every open set $H \subset Y$ containing $f(p)$ is a superset of an open set $G \subset X$ containing p or, equivalently, iff the inverse image of every neighborhood of $f(p)$ is a neighborhood of p, i.e.,

$$N \in \mathcal{N}_{f(p)} \quad \Rightarrow \quad f^{-1}[N] \in \mathcal{N}_p$$

Notice that, with respect to the usual topology on the real line $\mathbf{R}$, this definition coincides with the $\epsilon - \delta$ definition of continuity at a point for functions $f: \mathbf{R} \to \mathbf{R}$. In fact, the relationship between local and global continuity for functions $f: \mathbf{R} \to \mathbf{R}$ holds true in general; namely,

Theorem 7.5: Let X and Y be topological spaces. Then a function $f: X \to Y$ is continuous if and only if it is continuous at every point of X.

SEQUENTIAL CONTINUITY AT A POINT

A function $f: X \to Y$ is *sequentially continuous* at a point $p \in X$ iff for every sequence $\langle a_n \rangle$ in X converging to p, the sequence $\langle f(a_n) \rangle$ in Y converges to $f(p)$, i.e.,

$$a_n \to p \quad \text{implies} \quad f(a_n) \to f(p)$$

Sequential continuity and continuity at a point are related as follows:

Proposition 7.6: If a function $f: X \to Y$ is continuous at $p \in X$, then it is sequentially continuous at p.

Remark: The converse of the previous proposition is not true. Consider, for example, the topology $\mathcal{T}$ on the real line $\mathbf{R}$ consisting of $\emptyset$ and the complements of countable sets. Recall (see Example 7.3 of Chapter 5) that a sequence $\langle a_n \rangle$ converges to p if and only if it has the form

$$\langle a_1, a_2, \ldots, a_{n_0}, p, p, p, \ldots \rangle$$

Then for any function $f: (R, \mathcal{T}) \to (X, \mathcal{T}^*)$,

$$\langle f(a_n) \rangle \;=\; \langle f(a_1), \ldots, f(a_{n_0}), f(p), f(p), f(p), \ldots \rangle$$

converges to $f(p)$. In other words, every function on $(\mathbf{R}, \mathcal{T})$ is sequentially continuous. On the other hand, the function $f(\mathbf{R}, \mathcal{T}) \to (\mathbf{R}, \mathcal{U})$ defined by $f(x) = x$, i.e. the identity function, is not $\mathcal{T}$-$\mathcal{U}$ continuous since $f^{-1}[(0,1)] = (0,1)$ is not a $\mathcal{T}$-open subset of $\mathbf{R}$.

OPEN AND CLOSED FUNCTIONS

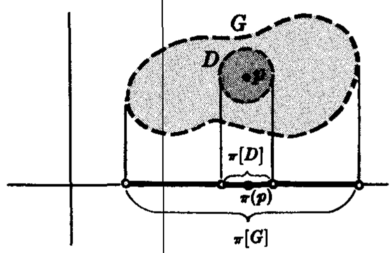
A continuous function has the property that the *inverse* image of every open set is open and the *inverse* image of every closed set is closed. It is natural then to ask about the following types of functions:

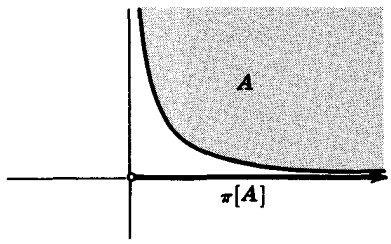
(1) A function $f: X \to Y$ is called an *open* (or *interior*) *function* if the image of every open set is open.

(2) A function $g: X \to Y$ is called a *closed function* if the image of every closed set is closed.

In general, functions which are open need not be closed and vice versa. In fact, the function in our first example is open and continuous but not closed.

Example 2.1: Consider the projection mapping $\pi : \mathbf{R}^2 \to \mathbf{R}$ of the plane $\mathbf{R}^2$ into the x-axis, i.e. $\pi(\langle x, y\rangle) = x$. Observe that the projection $\pi[D]$ of any open disc $D \subset \mathbf{R}^2$ is an open interval. Hence any point $\pi(p)$ in the image $\pi[G]$ of an open set $G \subset \mathbf{R}^2$ belongs to an open interval contained in $\pi[G]$, or $\pi[G]$ is open. Accordingly, π is an open function. On the other hand, π is not a closed function, for the set $A = \{\langle x, y\rangle : xy \geqq 1, x > 0\}$ is closed, but its projection $\pi[A] = (0, \infty)$ is not closed. (See diagrams below.)





HOMEOMORPHIC SPACES

A topological space $(X, \mathcal{T})$ is, as we have seen, a set X together with a distinguished class $\mathcal{T}$ of subsets of X, satisfying certain axioms. Between any two such spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}^*)$ there are many functions $f : X \to Y$. We choose to discuss continuous, or open, or closed functions rather than arbitrary functions since it is these functions which preserve some aspect of the structure of the spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}^*)$.

Now suppose there is some bijective (i.e. one-one and onto) mapping $f : X \to Y$. Then f induces a bijective function $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ from the power set of X, i.e. the class of subsets of X, into the power set of Y. If this induced function also takes $\mathcal{T}$ onto $\mathcal{T}^*$, i.e. defines a one-to-one correspondence between the open sets in X and the open sets in Y, then the spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}^*)$ are identical from the topological point of view. Specifically:

Definition: Two topological spaces X and Y are called *homeomorphic* or *topologically equivalent* if there exists a bijective (i.e. one-one, onto) function $f : X \to Y$ such that f and f^{-1} are continuous. The function f is called a *homeomorphism.*

A function f is called *bicontinuous* or *topological* if f is open and continuous. Thus $f : X \to Y$ is a homeomorphism iff f is bicontinuous and bijective.

Example 3.1: Let $X = (-1, 1)$. The function $f : X \to \mathbf{R}$ defined by $f(x) = \tan \frac{1}{2}\pi x$ is one-one, onto and continuous. Furthermore, the inverse function f^{-1} is also continuous. Hence the real line $\mathbf{R}$ and the open interval $(-1, 1)$ are homeomorphic.

Example 3.2: Let X and Y be discrete spaces. Then, as seen in Example 1.2, all functions from one to the other are continuous. Hence X and Y are homeomorphic iff there exists a one-one, onto function from one to the other, i.e. iff they are cardinally equivalent.

Proposition 7.7: The relation in any collection of topological spaces defined by "X is homeomorphic to Y" is an equivalence relation.

Thus, by the Fundamental Theorem on Equivalence Relations, any collection of topological spaces can be partitioned into classes of topologically equivalent spaces.

TOPOLOGICAL PROPERTIES

A property P of sets is called *topological* or a *topological invariant* if whenever a topological space $(X, \mathcal{T})$ has P then every space homeomorphic to $(X, \mathcal{T})$ also has P.

Example 4.1: As seen in Example 3.1, the real line **R** is homeomorphic to the open interval $X = (-1, 1)$. Hence *length* is not a topological property since X and **R** have different lengths, and *boundedness* is not a topological property since X is bounded but **R** is not.

Example 4.2: Let X be the set of positive real numbers, i.e. $X = (0, \infty)$. The function $f : X \to X$ defined by $f(x) = 1/x$ is a homeomorphism from X onto X. Observe that the sequence

$$\langle a_n \rangle \;=\; \langle 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots \rangle$$

corresponds, under the homeomorphism, to the sequence

$$\langle f(a_n) \rangle \;=\; \langle 1, 2, 3, \ldots \rangle$$

The sequence $\langle a_n \rangle$ is a Cauchy sequence; the sequence $\langle f(a_n) \rangle$ is not. Hence the property of being a Cauchy sequence is not topological.

Most of topology is an investigation of the consequences of certain topological properties as *compactness* and *connectedness*. In fact, formally topology is the study of topological invariants. In the next example, connectedness is defined and is shown to be a topological property.

Example 4.3: A topological space $(X, \mathcal{T})$ is *disconnected* iff X is the union of two open, non-empty, disjoint subsets, i.e.

$$X = G \cup H \quad \text{where} \quad G, H \in \mathcal{T}, \; G \cap H = \emptyset \quad \text{but} \quad G, H \neq \emptyset$$

If $f : X \to Y$ is a homeomorphism then $X = G \cup H$ if and only if $Y = f[G] \cup f[H]$ and so Y is disconnected if and only if X is.

The space $(X, \mathcal{T})$ is *connected* iff it is not disconnected.

TOPOLOGIES INDUCED BY FUNCTIONS

Let $\{(Y_i, \mathcal{T}_i)\}$ be any collection of topological spaces and for each Y_i let there be given a function $f_i : X \to Y_i$ defined on some arbitrary non-empty set X. We want to investigate those topologies on X with respect to which all the functions f_i are continuous. Recall that f_i is continuous relative to some topology on X provided the inverse image of each open subset of Y_i is an open subset of X. Thus we consider the following class of subsets of X:

$$\mathcal{S} \;=\; \textstyle\bigcup_i \{f_i^{-1}[H] : H \in \mathcal{T}_i\}$$

That is, $\mathcal{S}$ consists of the inverse image of each open subset of every space Y_i. The topology $\mathcal{T}$ on X generated by $\mathcal{S}$ is called the topology *induced* (or *generated*) by the functions f_i. The main properties of $\mathcal{T}$ are listed in the next theorem.

Theorem 7.8: (i) All the functions f_i are continuous relative to $\mathcal{T}$.

(ii) $\mathcal{T}$ is the intersection of all the topologies on X with respect to which the functions f_i are continuous.

(iii) $\mathcal{T}$ is the smallest, i.e. coarsest, topology on X with respect to which the functions f_i are continuous.

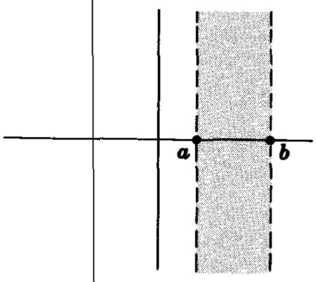
(iv) $\mathcal{S}$ is a subbase for the topology $\mathcal{T}$.

We shall call $\mathcal{S}$ the *defining subbase* for the topology induced by the functions f_i, i.e. the coarsest topology on X with respect to which the functions f_i are continuous.

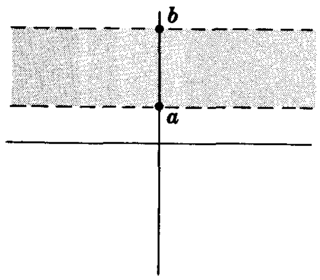
Example 5.1: Let π_1 and π_2 be the projections of the plane $\mathbf{R}^2$ into **R**, i.e.,

$$\pi_1(\langle x, y \rangle) = x \qquad \text{and} \qquad \pi_2(\langle x, y \rangle) = y$$

Observe, as illustrated below, that the inverse image of an open interval (a, b) in **R** is an infinite open strip in $\mathbf{R}^2$.



$\pi_1^{-1}[(a,b)]$



$\pi_2^{-1}[(a,b)]$

Recall that these infinite open strips form a subbase for the usual topology on $\mathbf{R}^2$. Accordingly, the usual topology on $\mathbf{R}^2$ is the smallest topology on $\mathbf{R}^2$ with respect to which the projections π_1 and π_2 are continuous.

Solved Problems

CONTINUOUS FUNCTIONS

1. Prove: Let $f: X \to Y$ be a constant function, say $f(x) = p \in Y$, for every $x \in X$. Then f is continuous relative to any topology $\mathcal{T}$ on X and any topology $\mathcal{T}^*$ on Y.

Solution:

We need to show that the inverse image of any $\mathcal{T}^*$-open subset of Y is a $\mathcal{T}$-open subset of X. Let $H \in \mathcal{T}^*$. Now $f(x) = p$ for all $x \in X$, so

$$f^{-1}[H] = \begin{cases} X & \text{if } p \in H \\ \emptyset & \text{if } p \notin H \end{cases}$$

In either case $f^{-1}[H]$ is an open subset of X since X and $\emptyset$ belong to every topology $\mathcal{T}$ on X.

2. Prove: Let $f: X \to Y$ be any function. If $(Y, \mathcal{J})$ is an indiscrete space, then $f: (X, \mathcal{T}) \to (Y, \mathcal{J})$ is continuous for any $\mathcal{T}$.

Solution:

We want to show that the inverse image of every open subset of Y is an open subset of X. Since $(Y, \mathcal{J})$ is an indiscrete space, Y and $\emptyset$ are the only open subsets of Y. But

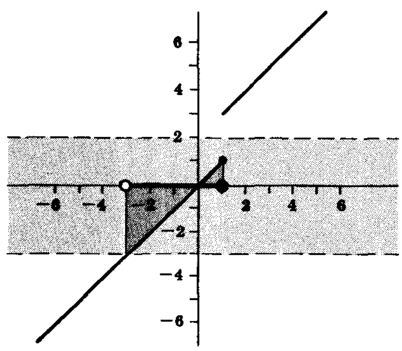
$$f^{-1}[Y] = X, \quad f^{-1}[\emptyset] = \emptyset$$

and X and $\emptyset$ belong to any topology $\mathcal{T}$ on X. Hence f is continuous for any $\mathcal{T}$.

3. Let $\mathcal{U}$ be the usual topology on the real line $\mathbf{R}$ and let $\mathcal{T}$ be the upper limit topology on $\mathbf{R}$ which is generated by the open-closed intervals $(a, b]$. Furthermore, let $f: \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x + 2 & \text{if } x > 1 \end{cases}$$

(See diagram on the right.)



(i) Show that f is not $\mathcal{U}$-$\mathcal{U}$ continuous.

(ii) Show that f is $\mathcal{T}$-$\mathcal{T}$ continuous.

Solution:

(i) Let $A = (-3, 2)$. Then $f^{-1}[A] = (-3, 1]$. Now $A \in \mathcal{U}$ but $f^{-1}[A] \notin \mathcal{U}$, so f is not $\mathcal{U}$-$\mathcal{U}$ continuous.

(ii) Let $A = (a, b]$. Then:

$$f^{-1}[A] = \begin{cases} (a, b] & \text{if } a < b \leq 1 \\ (a, 1] & \text{if } a < 1 < b \leq 3 \\ (a, b-2] & \text{if } a < 1 < 3 < b \\ \emptyset & \text{if } 1 \leq a < b \leq 3 \\ (1, b-2] & \text{if } 1 \leq a < 3 < b \\ (a-2, b-2] & \text{if } 3 \leq a < b \end{cases}$$

In each case, $f^{-1}[A]$ is a $\mathcal{T}$-open set. Hence f is $\mathcal{T}$-$\mathcal{T}$ continuous.

4. Suppose a function $f: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is not $\mathcal{T}_1$-$\mathcal{T}_2$ continuous. Show that if $\mathcal{T}_1^*$ is a topology on X coarser than $\mathcal{T}_1$ and if $\mathcal{T}_2^*$ is a topology on Y finer than $\mathcal{T}_2$, i.e. $\mathcal{T}_1^* \subset \mathcal{T}_1$ and $\mathcal{T}_2 \subset \mathcal{T}_2^*$, then f is also not $\mathcal{T}_1^*$-$\mathcal{T}_2^*$ continuous.

Solution:

Since $f: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is not continuous,

$$\exists G \in \mathcal{T}_2 \quad \text{for which} \quad f^{-1}[G] \notin \mathcal{T}_1$$

Now, $\mathcal{T}_1^* \subset \mathcal{T}_1$ and $\mathcal{T}_2 \subset \mathcal{T}_2^*$. Hence $G \in \mathcal{T}_2$ implies $G \in \mathcal{T}_2^*$, and $f^{-1}[G] \notin \mathcal{T}_1$ implies $f^{-1}[G] \notin \mathcal{T}_1^*$. Thus f is not continuous with respect to $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$.

5. Show that the identity function $i: (X, \mathcal{T}) \to (X, \mathcal{T}^*)$ is continuous if and only if $\mathcal{T}$ is finer than $\mathcal{T}^*$, i.e. $\mathcal{T}^* \subset \mathcal{T}$.

Solution:

By definition, i is $\mathcal{T}$-$\mathcal{T}^*$ continuous if and only if

$$G \in \mathcal{T}^* \quad \Rightarrow \quad i^{-1}[G] \in \mathcal{T}$$

But $i^{-1}[G] = G$, so i is $\mathcal{T}$-$\mathcal{T}^*$ continuous, if and only if

$$G \in \mathcal{T}^* \quad \Rightarrow \quad G \in \mathcal{T}$$

that is, $\mathcal{T}^* \subset \mathcal{T}$.

6. Prove Theorem 7.2: Let $f: (X, \mathcal{T}) \to (Y, \mathcal{T}^*)$, and let $\mathcal{S}$ be a subbase for the topology $\mathcal{T}^*$ on Y. Then f is continuous if and only if the inverse of every member of the subbase $\mathcal{S}$ is an open subset of X, i.e. $f^{-1}[S] \in \mathcal{T}$ for every $S \in \mathcal{S}$.

Solution:

Suppose $f^{-1}[S] \in \mathcal{T}$ for every $S \in \mathcal{S}$. We want to show that f is continuous, i.e. $G \in \mathcal{T}^*$ implies $f^{-1}[G] \in \mathcal{T}$. Let $G \in \mathcal{T}^*$. By definition of subbase,

$$G = \cup_i (S_{i_1} \cap \cdots \cap S_{i n_i}) \quad \text{where} \quad S_{i_k} \in \mathcal{S}$$

Hence,

$$f^{-1}[G] = f^{-1}[\cup_i (S_{i_1} \cap \cdots \cap S_{i n_i})] = \cup_i f^{-1}[S_{i_1} \cap \cdots \cap S_{i n_i}]$$
$$= \cup_i (f^{-1}[S_{i_1}] \cap \cdots \cap f^{-1}[S_{i n_i}])$$

But $S_{i_k} \in \mathcal{S}$ implies $f^{-1}[S_{i_k}] \in \mathcal{T}$. Hence $f^{-1}[G] \in \mathcal{T}$ since it is the union of finite intersections of open sets. Accordingly, f is continuous.

On the other hand, if f is continuous then the inverse of all open sets, including the members of $\mathcal{S}$ are open.

7. Let f be a function from a topological space X into the unit interval $[0, 1]$. Show that if $f^{-1}[(a, 1]]$ and $f^{-1}[[0, b)]$ are open subsets of X for all $0 < a, b < 1$, then f is continuous.

Solution:

Recall that the intervals $(a, 1]$ and $[0, b)$ form a subbase for the unit interval $I = [0, 1]$. Hence f is continuous by the preceding problem, i.e. by Theorem 7.2.

8. Prove: Let the functions $f: X \to Y$ and $g: Y \to Z$ be continuous. Then the composition function $g \circ f: X \to Z$ is also continuous.

Solution:

Let G be an open subset of Z. Then $g^{-1}[G]$ is open in Y since g is continuous. But f is also continuous, so $f^{-1}[g^{-1}[G]]$ is open in X. Now

$$(g \circ f)^{-1}[G] = f^{-1}[g^{-1}[G]]$$

Thus $(g \circ f)^{-1}[G]$ is open in X for every open subset G of Z, or, $g \circ f$ is continuous.

9. Prove: Let $\{\mathcal{T}_i\}$ be a collection of topologies on a set X. If a function $f: X \to Y$ is continuous with respect to each $\mathcal{T}_i$, then f is continuous with respect to the intersection topology $\mathcal{T} = \cap_i \mathcal{T}_i$.

Solution:

Let G be an open subset of Y. Then, by hypothesis, $f^{-1}[G]$ belongs to each $\mathcal{T}_i$. Hence $f^{-1}[G]$ belongs to the intersection, i.e. $f^{-1}[G] \in \cap_i \mathcal{T}_i = \mathcal{T}$, and so f is continuous with respect to $\mathcal{T}$.

10. Prove Theorem 7.3: A function $f: X \to Y$ is continuous if and only if the inverse image of every closed subset of Y is a closed subset of X.

Solution:

Suppose $f: X \to Y$ is continuous, and let F be a closed subset of Y. Then F^c is open, and so $f^{-1}[F^c]$ is open in X. But $f^{-1}[F^c] = (f^{-1}[F])^c$; therefore $f^{-1}[F]$ is closed.

Conversely, assume F closed in Y implies $f^{-1}[F]$ closed in X. Let G be an open subset of Y. Then G^c is closed in Y, and so $f^{-1}[G^c] = (f^{-1}[G])^c$ is closed in X. Accordingly, $f^{-1}[G]$ is open and therefore f is continuous.

11. Prove Theorem 7.4: A function $f: X \to Y$ is continuous if and only if, for every subset $A \subset X$, $f[\bar{A}] \subset \overline{f[A]}$.

Solution:

Suppose $f: X \to Y$ is continuous. Now $f[A] \subset \overline{f[A]}$, so

$$A \subset f^{-1}[f[A]] \subset f^{-1}[\overline{f[A]}]$$

But $\overline{f[A]}$ is closed, and so $f^{-1}[\overline{f[A]}]$ is also closed; hence

$$A \subset \bar{A} \subset f^{-1}[\overline{f[A]}]$$

and therefore

$$f[\bar{A}] \subset \overline{f[A]} = f[f^{-1}[\overline{f[A]}]]$$

Conversely, assume $f[\bar{A}] \subset \overline{f[A]}$ for any $A \subset X$, and let F be a closed subset of Y. Set $A = f^{-1}[F]$; we wish to show that A is also closed or, equivalently, that $\bar{A} = A$. Now

$$f[\bar{A}] = f[\overline{f^{-1}[F]}] \subset \overline{f[f^{-1}[F]]} = \bar{F} = F$$

Hence

$$\bar{A} \subset f^{-1}[f[\bar{A}]] \subset f^{-1}[F] = A$$

But $A \subset \bar{A}$, so $\bar{A} = A$ and f is continuous.

12. Prove: Let $f: (X, \mathcal{T}) \to (Y, \mathcal{T}^*)$ be continuous. Then $f_A: (A, \mathcal{T}_A) \to (Y, \mathcal{T}^*)$ is continuous, where $A \subset X$ and f_A is the restriction of f to A.

Solution:

Observe that $f_A^{-1}[G] = A \cap f^{-1}[G]$ for any $G \subset Y$.

Let $G \in \mathcal{T}^*$. Then $f^{-1}[G] \in \mathcal{T}$, and so $A \cap f^{-1}[G] \in \mathcal{T}_A$ by definition of the induced topology. Thus $A \cap f^{-1}[G] = f_A^{-1}[G] \in \mathcal{T}_A$, so f_A is continuous.

CONTINUITY AT A POINT

13. Under what conditions will a function $f: X \to Y$ not be continuous at a point $p \in X$?

Solution:

A function $f: X \to Y$ is continuous at $p \in X$ iff, for every open set $H \subset Y$ containing $f(p)$, $f^{-1}[H]$ is a superset of an open set containing p. Hence f is not continuous at $p \in X$ if there exists at least one open set $H \subset Y$ containing $f(p)$ such that $f^{-1}[H]$ does not contain an open set containing p.

Equivalently, $f: X \to Y$ is not continuous at $p \in X$ iff $\exists$ a neighborhood N of $f(p)$ such that $f^{-1}[N]$ is not a neighborhood of p.

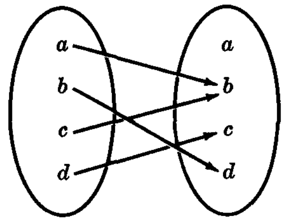
14. Consider the following topology defined on $X = \{a, b, c, d\}$:

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$$

Let the function $f: X \to X$ be defined by the adjoining diagram.

(i) Show that f is not continuous at c.

(ii) Show that f is continuous at d.



Solution:

(i) Observe that $\{a, b\}$ is an open set containing $f(c) = b$ and that $f^{-1}[\{a, b\}] = \{a, c\}$. Hence f is not continuous at c since there exists no open set containing c which is contained in $\{a, c\}$.

(ii) The only open sets containing $f(d) = c$ are $\{b, c, d\}$ and X. Note that $f^{-1}[\{b, c, d\}] = X$ and $f^{-1}[X] = X$. Hence f is continuous at d since the inverse of each open set containing $f(d)$ is an open set containing d.

15. Suppose a singleton set $\{p\}$ is an open subset of a topological space X. Show that for any topological space Y and any function $f: X \to Y$, f is continuous at $p \in X$.

Solution:

Let $H \subset Y$ be an open set containing $f(p)$. But

$$f(p) \in H \quad \Rightarrow \quad p \in f^{-1}[H] \quad \Rightarrow \quad \{p\} \subset f^{-1}[H]$$

Hence f is continuous at p.

16. Prove: If $f: X \to Y$ is continuous at $p \in X$, then the restriction of f to a subset containing p is also continuous at p. More precisely, let A be a subset of a topological space $(X, \mathcal{T})$ such that $p \in A \subset X$, and let $f_A: A \to Y$ denote the restriction of $f: X \to Y$ to A. Then if f is $\mathcal{T}$-continuous at p, f_A will be $\mathcal{T}_A$-continuous at p where $\mathcal{T}_A$ is the relative topology on A.

Solution:

Let $H \subset Y$ be an open set containing $f(p)$. Since f is continuous at p,

$$\exists G \in \mathcal{T} \quad \text{such that} \quad p \in G \subset f^{-1}[H]$$

and so

$$p \in A \cap G \subset A \cap f^{-1}[H] = f_A^{-1}[H]$$

But, by definition of the induced topology, $A \cap G \in \mathcal{T}_A$; hence f_A is $\mathcal{T}_A$-continuous at p.

17. Prove Theorem 7.5: Let X and Y be topological spaces. Then a function $f: X \to Y$ is continuous if and only if it is continuous at every point $p \in X$.

Solution:

Assume f is continuous, and let $H \subset Y$ be an open set containing $f(p)$. But then $p \in f^{-1}[H]$, and $f^{-1}[H]$ is open. Hence f is continuous at p.

Now suppose f is continuous at every point $p \in X$, and let $H \subset Y$ be open. For every $p \in f^{-1}[H]$, there exists an open set $G_p \subset X$ such that $p \in G_p \subset f^{-1}[H]$. Hence $f^{-1}[H] = \bigcup\{G_p : p \in f^{-1}[H]\}$ a union of open sets. Accordingly, $f^{-1}[H]$ is open and so f is continuous.

18. Prove Proposition 7.6: If a function $f: X \to Y$ is continuous at $p \in X$, then it is sequentially continuous at p, i.e. $a_n \to p \Rightarrow f(a_n) \to f(p)$.

Solution:

We need to show that any neighborhood N of $f(p)$ contains almost all the terms of the sequence $\langle f(a_1), f(a_2), \ldots \rangle$.

Let N be a neighborhood of $f(p)$. By hypothesis, f is continuous at p; hence $M = f^{-1}[N]$ is a neighborhood of p. If the sequence $\langle a_n \rangle$ converges to p, then M contains almost all the terms of the sequence $\langle a_1, a_2, \ldots \rangle$, i.e. $a_n \in M$ for almost all $n \in N$. But

$$a_n \in M \quad \Rightarrow \quad f(a_n) \in f[M] = f[f^{-1}[N]] = N$$

Hence $f(a_n) \in N$ for almost all $n \in N$, and so the sequence $\langle f(a_n) \rangle$ converges to $f(p)$. Accordingly, f is sequentially continuous at p.

OPEN AND CLOSED FUNCTIONS, HOMEOMORPHISMS

19. Give an example of a real function $f: \mathbf{R} \to \mathbf{R}$ such that f is continuous and closed, but not open.

Solution:

Let f be a constant function, say $f(x) = 1$ for all $x \in \mathbf{R}$. Then $f[A] = \{1\}$ for any $A \subset \mathbf{R}$. Hence f is a closed function and is not an open function. Furthermore, f is continuous.

20. Let the real function $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$. Show that f is not open.

Solution:

Let $A = (-1, 1)$, an open set. Note that $f[A] = [0, 1)$, which is not open; hence f is not an open function.

21. Let $\mathcal{B}$ be a base for a topological space X. Show that if $f: X \to Y$ has the property that $f[B]$ is open for every $B \in \mathcal{B}$, then f is an open function.

Solution:

We want to show that the image of every open subset of X is open in Y. Let $G \subset X$ be open. By definition of a base, $G = \cup_i B_i$ where $B_i \in \mathcal{B}$. Now $f[G] = f[\cup_i B_i] = \cup_i f[B_i]$. By hypothesis, each $f[B_i]$ is open in Y and so $f[G]$, a union of open sets, is also open in Y; hence f is an open function.

22. Show that the closed interval $A = [a, b]$ is homeomorphic to the closed unit interval $I = [0, 1]$.

Solution:

The linear function $f: I \to A$ defined by $f(x) = (b-a)x + a$ is one-one, onto and bicontinuous. Hence f is a homeomorphism.

23. Show that area is not a topological property.

Solution:

The open disc $D = \{\langle r, \theta \rangle : r < 1\}$ with radius 1 is homeomorphic to the open disc $D^* = \{\langle r, \theta \rangle : r < 2\}$ with radius 2. In fact, the function $f: D \to D^*$ defined by $f(\langle r, \theta \rangle) = \langle 2r, \theta \rangle$ is a homeomorphism. Here $\langle r, \theta \rangle$ denotes the polar coordinates of a point in the plane $\mathbf{R}^2$.

24. Let $f: (X, \mathcal{T}) \to (Y, \mathcal{T}^*)$ be one-one and open, let $A \subset X$, and let $f[A] = B$. Show that the function $f_A: (A, \mathcal{T}_A) \to (B, \mathcal{T}^*_B)$ is also one-one and open. Here f_A denotes the restriction of f to A, and $\mathcal{T}_A$ and $\mathcal{T}^*_B$ are the relative topologies.

Solution:

If f is one-one, then every restriction of f is also one-one; hence we need only show that f_A is open.

Let $H \subset A$ be $\mathcal{T}_A$-open. Then by definition of the relative topology, $H = A \cap G$ where $G \in \mathcal{T}$. Since f is one-one, $f[A \cap G] = f[A] \cap f[G]$, and so

$$f_A[H] = f[H] = f[A \cap G] = f[A] \cap f[G] = B \cap f[G]$$

Since f is open and $G \in \mathcal{T}$, $f[G] \in \mathcal{T}^*$. Thus $B \cap f[G] \in \mathcal{T}_B^*$ and so $f_A : (A, \mathcal{T}_A) \to (B, \mathcal{T}_B^*)$ is open.

25. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}^*)$ be a homeomorphism and let $(A, \mathcal{T}_A)$ be any subspace of $(X, \mathcal{T})$. Show that $f_A : (A, \mathcal{T}_A) \to (B, \mathcal{T}_B^*)$ is also a homeomorphism where f_A is the restriction of f to A, $f[A] = B$, and $\mathcal{T}_B^*$ is the relative topology on B.

Solution:

Since f is one-one and onto, $f_A : A \to B$, where $B = f[A]$, is also one-one and onto. Hence we need only show that f_A is bicontinuous, i.e. open and continuous. By the preceding problem f_A is open. Furthermore, the restriction of any continuous function is also continuous; hence $f_A : (A, \mathcal{T}_A) \to (B, \mathcal{T}_B^*)$ is a homeomorphism.

26. Show that any interval $A = (a, b)$ is connected as a subspace of the real line $\mathbf{R}$. (See Example 4.3 for the definition of connectedness.)

Solution:

Suppose A is not connected. Then $\exists$ open sets $G, H \subset \mathbf{R}$ such that $A \cap G$ and $A \cap H$ are non-empty, disjoint and satisfy $(A \cap G) \cup (A \cap H) = A$. Define the function $f : A \to \mathbf{R}$ by

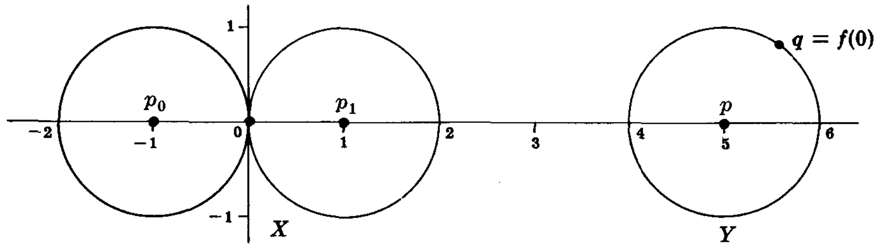
$$f(x) = \begin{cases} 1 & \text{if } x \in A \cap G \\ 0 & \text{if } x \in A \cap H \end{cases}$$

Then f is continuous, for the inverse of any open set is either $A \cap G$, $A \cap H$, $\emptyset$ or A and so is open. But then the intermediate value theorem applies, so $\exists x_0 \in A$ for which $f(x_0) = \frac{1}{2}$. But this is impossible, so A is connected.

27. Show that the following subsets of the plane $\mathbf{R}^2$ are not homeomorphic, where the topologies are the relativized usual topologies:

$$X = \{x : d(x, p_0) = 1 \text{ or } d(x, p_1) = 1;\ p_0 = \langle 0, -1 \rangle,\ p_1 = \langle 0, 1 \rangle\}$$

$$Y = \{x : d(x, p) = 1,\ p = \langle 0, 5 \rangle\}$$



Solution:

Suppose there exists a homeomorphism $f : X \to Y$; let $q = f(0)$, $X^* = X \setminus \{0\}$, and $Y^* = Y \setminus \{q\}$. Then $f : X^* \to Y^*$ is also a homeomorphism with respect to the relative topologies (see Problem 25).

We show that Y^* is connected. For if $q = \langle 5 + \cos \theta_0, \sin \theta_0 \rangle$, then the function

$$g : (0, 2\pi) \to Y^* \quad \text{defined by} \quad g(\theta) = \langle 5 + \cos(\theta_0 + \theta), \sin(\theta_0 + \theta) \rangle$$

is a homeomorphism. But the interval $(0, 2\pi)$ is connected, so Y^* is also connected.

On the other hand, X^* is not connected; for the sets

$$G = \{\langle x, y \rangle : x > 0\} \quad \text{and} \quad H = \{\langle x, y \rangle : x < 0\}$$

are both open in $\mathbf{R}^2$, so $G^* = X^* \cap G$ and $H^* = X^* \cap H$ are open subsets of X^*. Furthermore, G^* and H^* are non-empty, disjoint and satisfy $G^* \cup H^* = X^*$. Since connectedness is a topological property, X^* is not homeomorphic to Y^* and therefore there can exist no such function f.

TOPOLOGIES INDUCED BY FUNCTIONS

28. Let $\{f_i : X \to (Y_i, \mathcal{T}_i)\}$ be a collection of constant functions from an arbitrary set X into the topological spaces $(Y_i, \mathcal{T}_i)$. Determine the coarsest topology on X with respect to which the functions f_i are continuous.

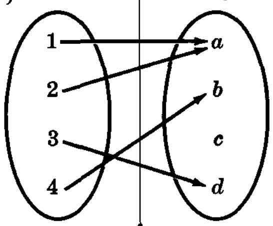
Solution:

Recall (see Problem 1) that a constant function $f : X \to Y$ is continuous with respect to every topology on X. Hence all the constant functions f_i are continuous with respect to the indiscrete topology $\{X, \emptyset\}$ on X. Since the indiscrete topology $\{X, \emptyset\}$ on X is the coarsest topology on X, it is also the coarsest topology on X with respect to which the constant functions are continuous.

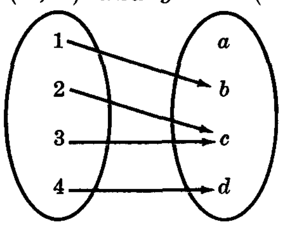
29. Consider the following topology on $Y = \{a, b, c, d\}$:

$$\mathcal{T} = \{Y, \emptyset, \{c\}, \{a,b,c\}, \{c,d\}\}$$

Let $X = \{1,2,3,4\}$ and let the functions $f : X \to (Y, \mathcal{T})$ and $g : X \to (Y, \mathcal{T})$ be defined by



f



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Find the defining subbase $\mathcal{S}$ for the topology $\mathcal{T}^*$ on X induced by f and g, i.e. the coarsest topology with respect to which f and g are continuous.

Solution:

Recall that
$$\mathcal{S} = \{f^{-1}[H] : H \in \mathcal{T}\} \cup \{g^{-1}[H] : H \in \mathcal{T}\}$$
that is, $\mathcal{S}$ consists of the inverses under f and g of the open subsets of Y. Hence
$$\mathcal{S} = \{X, \emptyset, \{1,2,4\}, \{3\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}\}$$

30. Let $\mathcal{T}$ be the topology on the real line $\mathbf{R}$ generated by the closed-open intervals $[a, b)$, and let $\mathcal{T}^*$ be the topology on $\mathbf{R}$ induced by the collection of all linear functions

$$f : \mathbf{R} \to (\mathbf{R}, \mathcal{T}) \quad \text{defined by} \quad f(x) = ax + b, \quad a, b \in \mathbf{R}$$

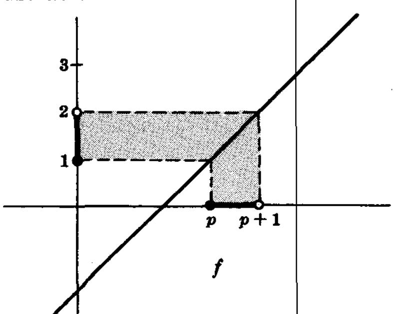
Show that $\mathcal{T}^*$ is the discrete topology on $\mathbf{R}$.

Solution:

We want to show that, for every $p \in \mathbf{R}$, the singleton set $\{p\}$ is a $\mathcal{T}^*$-open set. Consider the $\mathcal{T}$-open set $A = [1, 2)$ and the functions $f : \mathbf{R} \to (\mathbf{R}, \mathcal{T})$ and $g : \mathbf{R} \to (\mathbf{R}, \mathcal{T})$ defined by

$$f(x) = x - p + 1 \quad \text{and} \quad g(x) = -x - p + 1$$

and illustrated below.



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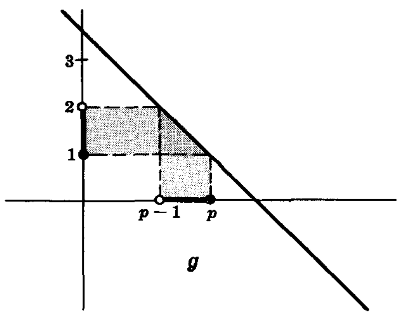

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Now $A \in \mathcal{T}$ implies

$$f^{-1}[A] = [p, p+1) \quad \text{and} \quad g^{-1}[A] = (p-1, p]$$

belong to the defining subbase $\mathcal{S}$ for the topology $\mathcal{T}^*$. Hence the intersection

$$(p-1, p] \cap [p, p+1) = \{p\}$$

belongs to $\mathcal{T}^*$, and so $\mathcal{T}^*$ is the discrete topology on **R**.

31. Prove Theorem 7.9: Let $\{f_i : X \to (Y_i, \mathcal{T}_i)\}$ be a collection of functions defined on an arbitrary non-empty set X, let

$$\mathcal{S} = \bigcup_i \{f_i^{-1}[H] : H \in \mathcal{T}_i\}$$

and let $\mathcal{T}$ be the topology on X generated by $\mathcal{S}$. Then:

(i) All the functions f_i are continuous relative to $\mathcal{T}$.

(ii) If $\mathcal{T}^*$ is the intersection of all topologies on X with respect to which the functions f_i are continuous, then $\mathcal{T} = \mathcal{T}^*$.

(iii) $\mathcal{T}$ is the coarsest topology on X with respect to which the functions f_i are continuous.

(iv) $\mathcal{S}$ is a subbase for $\mathcal{T}$.

Solution:

(i) For any function $f_i : (X, \mathcal{T}) \to (Y_i, \mathcal{T}_i)$, if $H \in \mathcal{T}_i$ then $f_i^{-1}[H] \in \mathcal{S} \subset \mathcal{T}$. Hence all the f_i are continuous with respect to $\mathcal{T}$.

(ii) By Problem 9, all the functions f_i are also continuous with respect to $\mathcal{T}^*$; hence $\mathcal{S} \subset \mathcal{T}^*$ and, since $\mathcal{T}$ is the topology generated by $\mathcal{S}$, $\mathcal{T} \subset \mathcal{T}^*$. On the other hand, $\mathcal{T}$ is one of the topologies with respect to which the f_i are continuous; hence $\mathcal{T}^* \subset \mathcal{T}$ and so $\mathcal{T} = \mathcal{T}^*$.

(iii) Follows from (ii).

(iv) Follows from the fact that any class of sets is a subbase of the topology it generates.

Supplementary Problems

CONTINUOUS FUNCTIONS

32. Prove that $f : X \to Y$ is continuous if and only if $f^{-1}[A^\circ] \subset (f^{-1}[A])^\circ$ for every $A \subset X$.

33. Let X and Y be topological spaces with $X = E \cup F$. Let $f : E \to Y$ and $g : F \to Y$, with $f = g$ on $E \cap F$, be continuous with respect to the relative topologies. Note that $h = f \cup g$ is a function from X into Y. (i) Show, by an example, that h need not be continuous. (ii) Prove: If E and F are both open, then h is continuous. (iii) Prove: If E and F are both closed, then h is continuous.

34. Let $f : X \to Y$ be continuous. Show that $f : X \to f[X]$ is also continuous where $f[X]$ has the relative topology.

35. Let X be a topological space and let $\chi_A : X \to \mathbf{R}$ be the characteristic function for some subset A of X. Show that χ_A is continuous at $p \in X$, if and only if p is not an element of the boundary of A. (Recall $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ if $x \in A^c$.)

36. Consider the real line **R** with the usual topology. Show that if every function $f : X \to \mathbf{R}$ is continuous, then X is a discrete space.

OPEN AND CLOSED FUNCTIONS

37. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}^*)$. Prove the following:

(i) f is closed if and only if $\overline{f[A]} \subset f[\bar{A}]$ for every $A \subset X$;

(ii) f is open if and only if $(f[A])^\circ \subset f[A^\circ]$ for every $A \subset X$.

38. Show that the function $f:(0,\infty)\to[-1,1]$ defined by $f(x)=\sin(1/x)$ is continuous, but neither open nor closed, where $(0,\infty)$ and $[-1,1]$ have the relativized usual topologies.

39. Prove: Let $f:(X,\mathcal{T})\to(Y,\mathcal{T}^*)$ be open and onto, and let $\mathcal{B}$ be a base for $\mathcal{T}$. Then $\{f[B]: B\in\mathcal{B}\}$ is a base for $\mathcal{T}^*$.

40. Give an example of a function $f:X\to Y$ and a subset $A\subset X$ such that f is open but f_A, the restriction of f to A, is not open.

HOMEOMORPHISMS, TOPOLOGICAL PROPERTIES

41. Let $f:X\to Y$ and $g:Y\to Z$ be continuous. Show that if $g\circ f: X\to Z$ is a homeomorphism, then g one-one (or f onto) implies that f and g are homeomorphisms.

42. Prove that each of the following is a topological property: (i) accumulation point, (ii) interior, (iii) boundary, (iv) density, and (v) neighborhood.

43. Prove: Let $f:X\to Y$ be a homeomorphism and let $A\subset X$ have the property that $A\cap A'=\emptyset$. Then $f[A]\cap(f[A])'=\emptyset$. (A subset $A\subset X$ having the property $A\cap A'=\emptyset$ is called *isolated*. The property of being isolated is thus a topological property.)

TOPOLOGIES INDUCED BY FUNCTIONS

44. Consider the following topology on $Y=\{a,b,c,d\}$: $\mathcal{T}=\{Y,\emptyset,\{a,b\},\{c,d\}\}$. Let $X=\{1,2,3,4,5\}$ and let $f:X\to Y$ and $g:X\to Y$ be as follows:

$$f=\{\langle 1,a\rangle,\langle 2,a\rangle,\langle 3,b\rangle,\langle 4,b\rangle,\langle 5,d\rangle\},\qquad g=\{\langle 1,c\rangle,\langle 2,b\rangle,\langle 3,d\rangle,\langle 4,a\rangle,\langle 5,c\rangle\}$$

Find the defining subbase for the topology on X induced by f and g.

45. Let $f:X\to(Y,\mathcal{T}^*)$. Show that if $\mathcal{S}$ is the defining subbase for the topology $\mathcal{T}$ induced by the one function f, then $\mathcal{S}=\mathcal{T}$.

46. Prove: Let $\{f_i: X\to(Y_i,\mathcal{T}_i)\}$ be a collection of functions defined on an arbitrary set X, and let $\mathcal{S}_i$ be a subbase for the topology $\mathcal{T}_i$ on Y_i. Then the class $\mathcal{S}^*=\bigcup_i\{f_i^{-1}[S]: S\in\mathcal{S}_i\}$ has the following properties: (i) $\mathcal{S}^*$ is a subclass of the defining subbase $\mathcal{S}$ of the topology $\mathcal{T}$ on X induced by the functions f_i; (ii) $\mathcal{S}^*$ is also a subbase for $\mathcal{T}$.

47. Show that the coarsest topology on the real line $\mathbf{R}$ with respect to which the linear functions

$$f:\mathbf{R}\to(\mathbf{R},\mathcal{U})\quad\text{defined by}\quad f(x)=ax+b,\quad a,b\in\mathbf{R}$$

are continuous is also the usual topology $\mathcal{U}$.

Answers to Supplementary Problems

33. (i) Let $X=(0,2)$ and let $E=(0,1)$ and $F=[1,2)$. Then $f(x)=1$ and $g(x)=2$ are each continuous, but $h=f\cup g$ is not continuous.

44. $\{X,\emptyset,\{1,2,3,4\},\{5\},\{2,4\},\{1,3,5\}\}$

45. *Hint.* Show that $\mathcal{S}$ is a topology.