Chapter 5

Topological Spaces: Definitions

TOPOLOGICAL SPACES

Let X be a non-empty set. A class $\mathcal{T}$ of subsets of X is a *topology* on X iff $\mathcal{T}$ satisfies the following axioms.

[O_1] X and $\emptyset$ belong to $\mathcal{T}$.

[O_2] The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

[O_3] The intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The members of $\mathcal{T}$ are then called *$\mathcal{T}$-open sets,* or simply *open sets,* and X together with $\mathcal{T}$, i.e. the pair $(X, \mathcal{T})$ is called a *topological space.*

Example 1.1: Let $\mathcal{U}$ denote the class of all open sets of real numbers discussed in Chapter 4. Then $\mathcal{U}$ is a topology on $\mathbf{R}$; it is called the *usual topology* on $\mathbf{R}$. Similarly, the class $\mathcal{U}$ of all open sets in the plane $\mathbf{R}^2$ is a topology, and also called the *usual topology*, on $\mathbf{R}^2$. We shall always assume the usual topology on $\mathbf{R}$ and $\mathbf{R}^2$ unless otherwise specified.

Example 1.2: Consider the following classes of subsets of $X = \{a, b, c, d, e\}$.

$$\mathcal{T}_1 = \{X,\ \emptyset,\ \{a\},\ \{c,d\},\ \{a,c,d\},\ \{b,c,d,e\}\}$$
$$\mathcal{T}_2 = \{X,\ \emptyset,\ \{a\},\ \{c,d\},\ \{a,c,d\},\ \{b,c,d\}\}$$
$$\mathcal{T}_3 = \{X,\ \emptyset,\ \{a\},\ \{c,d\},\ \{a,c,d\},\ \{a,b,d,e\}\}$$

Observe that $\mathcal{T}_1$ is a topology on X since it satisfies the necessary three axioms $[O_1]$, $[O_2]$ and $[O_3]$. But $\mathcal{T}_2$ is not a topology on X since the union

$$\{a,c,d\} \cup \{b,c,d\} = \{a,b,c,d\}$$

of two members of $\mathcal{T}_2$ does not belong to $\mathcal{T}_2$, i.e. $\mathcal{T}_2$ does not satisfy the axiom $[O_2]$.

Also, $\mathcal{T}_3$ is not a topology on X since the intersection

$$\{a,c,d\} \cap \{a,b,d,e\} = \{a,d\}$$

of two sets in $\mathcal{T}_3$ does not belong to $\mathcal{T}_3$, i.e. $\mathcal{T}_3$ does not satisfy the axiom $[O_3]$.

Example 1.3: Let $\mathcal{D}$ denote the class of all subsets of X. Observe that $\mathcal{D}$ satisfies the axioms for a topology on X. This topology is called the *discrete topology*; and X together with its discrete topology, i.e. the pair $(X, \mathcal{D})$, is called a *discrete topological space* or simply a *discrete space.*

Example 1.4: As seen by axiom $[O_1]$, a topology on X must contain the sets X and $\emptyset$. The class $\mathcal{J} = \{X, \emptyset\}$, consisting of X and $\emptyset$ alone, is itself a topology on X. It is called the *indiscrete topology*; and X together with its indiscrete topology, i.e. $(X, \mathcal{J})$, is called an *indiscrete topological space* or simply an *indiscrete space.*

Example 1.5: Let $\mathcal{T}$ denote the class of all subsets of X whose complements are finite together with the empty set $\emptyset$. This class $\mathcal{T}$ is also a topology on X. It is called the *cofinite topology* or the *T_1-topology* on X. (The significance of the T_1 will appear in a later chapter.)

Example 1.6: The intersection $\mathcal{T}_1 \cap \mathcal{T}_2$ of any two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X is also a topology on X. For, by $[\mathbf{O}_1]$, X and $\emptyset$ each belongs to both $\mathcal{T}_1$ and $\mathcal{T}_2$; hence X and $\emptyset$ each belongs to the intersection $\mathcal{T}_1 \cap \mathcal{T}_2$, i.e. $\mathcal{T}_1 \cap \mathcal{T}_2$ satisfies $[\mathbf{O}_1]$. Furthermore, if $G,H \in \mathcal{T}_1 \cap \mathcal{T}_2$ then, in particular, $G,H \in \mathcal{T}_1$ and $G,H \in \mathcal{T}_2$. But since $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies, $G \cap H \in \mathcal{T}_1$ and $G \cap H \in \mathcal{T}_2$. Accordingly,

$$G \cap H \in \mathcal{T}_1 \cap \mathcal{T}_2$$

In other words $\mathcal{T}_1 \cap \mathcal{T}_2$ satisfies $[\mathbf{O}_3]$. Similarly, $\mathcal{T}_1 \cap \mathcal{T}_2$ satisfies $[\mathbf{O}_2]$.

The statement in the preceding example can, in fact, be generalized to any collection of topologies. Namely,

Theorem 5.1: Let $\{\mathcal{T}_i : i \in I\}$ be any collection of topologies on a set X. Then the intersection $\cap_i \mathcal{T}_i$ is also a topology on X.

In our last example, we show that the union of topologies need not be a topology.

Example 1.7: Each of the classes

$$\mathcal{T}_1 = \{X, \emptyset, \{a\}\} \quad \text{and} \quad \mathcal{T}_2 = \{X, \emptyset, \{b\}\}$$

is a topology on $X = \{a, b, c\}$. But the union

$$\mathcal{T}_1 \cup \mathcal{T}_2 = \{X, \emptyset, \{a\}, \{b\}\}$$

is not a topology on X since it violates $[\mathbf{O}_2]$. That is, $\{a\} \in \mathcal{T}_1 \cup \mathcal{T}_2$, $\{b\} \in \mathcal{T}_1 \cup \mathcal{T}_2$ but $\{a\} \cup \{b\} = \{a, b\}$ does not belong to $\mathcal{T}_1 \cup \mathcal{T}_2$.

If G is an open set containing a point $p \in X$, then G is called an *open neighborhood* of p. Also, G without p, i.e. $G \setminus \{p\}$, is called a *deleted open neighborhood* of p.

Remark: The axioms $[\mathbf{O}_1]$, $[\mathbf{O}_2]$ and $[\mathbf{O}_3]$ are equivalent to the following two axioms:

$[\mathbf{O}_1^*]$ The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

$[\mathbf{O}_2^*]$ The intersection of any finite number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

For $[\mathbf{O}_1^*]$ implies that $\emptyset$ belongs to $\mathcal{T}$ since

$$\cup\{G \in \mathcal{T} : G \in \emptyset\} = \emptyset$$

i.e. the empty union of sets is the empty set. Furthermore, $[\mathbf{O}_2^*]$ implies that X belongs to $\mathcal{T}$ since

$$\cap\{G \in \mathcal{T} : G \in \emptyset\} = X$$

i.e. the empty intersection of subsets of X is X itself.

ACCUMULATION POINTS

Let X be a topological space. A point $p \in X$ is an *accumulation point* or *limit point* (also called *cluster point* or *derived point*) of a subset A of X iff every open set G containing p contains a point of A different from p, i.e.,

$$G \text{ open}, \ p \in G \quad \text{implies} \quad (G \setminus \{p\}) \cap A \neq \emptyset$$

The set of accumulation points of A, denoted by A', is called the *derived set* of A.

Example 2.1: The class $\mathcal{T} = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\, e\}\}$ defines a topology on $X = \{a, b, c, d, e\}$. Consider the subset $A = \{a, b, c\}$ of X. Observe that $b \in X$ is a limit point of A since the open sets containing b are $\{b, c, d, e\}$ and X, and each contains a point of A different from b, i.e. c. On the other hand, the point $a \in X$ is not a limit point of A since the open set $\{a\}$, which contains a, does not contain a point of A different from a. Similarly, the points d and e are limit points of A and the point c is not a limit point of A. So $A' = \{b, d, e\}$ is the derived set of A.

Example 2.2: Let X be an indiscrete topological space, i.e. X and $\emptyset$ are the only open subsets of X. Then X is the only open set containing any point $p \in X$. Hence p is an accumulation point of every subset of X except the empty set $\emptyset$ and the set consisting of p alone, i.e. the singleton set $\{p\}$. Accordingly, the derived set A' of any subset A of X is as follows:

$$A' = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \{p\}^c = X \setminus \{p\} & \text{if } A = \{p\} \\ X & \text{if } A \text{ contains two or more points} \end{cases}$$

Observe that, for the usual topology on the line $\mathbf{R}$ and the plane $\mathbf{R}^2$, the above definition of an accumulation point is the same as that given in Chapter 4.

CLOSED SETS

Let X be a topological space. A subset A of X is a *closed set* iff its complement A^c is an open set.

Example 3.1: The class $\mathcal{T} = \{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$ defines a topology on $X = \{a,b,c,d,e\}$. The closed subsets of X are

$$\emptyset,\ X,\ \{b,c,d,e\},\ \{a,b,e\},\ \{b,e\},\ \{a\}$$

that is, the complements of the open subsets of X. Note that there are subsets of X, such as $\{b,c,d,e\}$, which are both open and closed, and there are subsets of X, such as $\{a,b\}$, which are neither open nor closed.

Example 3.2: Let X be a discrete topological space, i.e. every subset of X is open. Then every subset of X is also closed since its complement is always open. In other words, all subsets of X are both open and closed.

Recall that $A^{cc} = A$, for any subset A of a space X. Hence

Proposition 5.2: In a topological space X, a subset A of X is open if and only if its complement is closed.

The axioms $[\mathbf{O}_1]$, $[\mathbf{O}_2]$ and $[\mathbf{O}_3]$ of a topological space and DeMorgan's Laws give

Theorem 5.3: Let X be a topological space. Then the class of closed subsets of X possesses the following properties:

(i) X and $\emptyset$ are closed sets.

(ii) The intersection of any number of closed sets is closed.

(iii) The union of any two closed sets is closed.

Closed sets can also be characterized in terms of their limit points as follows:

Theorem 5.4: A subset A of a topological space X is closed if and only if A contains each of its accumulation points.

In other words, a set A is closed if and only if the derived set A' of A is a subset of A, i.e. $A' \subset A$.

CLOSURE OF A SET

Let A be a subset of a topological space X. The *closure of A*, denoted by

$$\bar{A} \quad \text{or} \quad A^-$$

is the intersection of all closed supersets of A. In other words, if $\{F_i : i \in I\}$ is the class of all closed subsets of X containing A, then

$$\bar{A} = \cap_i F_i$$

Observe first that $\bar{A}$ is a closed set since it is the intersection of closed sets. Furthermore, $\bar{A}$ is the smallest closed superset of A, that is, if F is a closed set containing A, then

$$A \subset \bar{A} \subset F$$

Accordingly, a set A is closed if and only if $A = \bar{A}$. We state these results formally:

Proposition 5.5: Let $\bar{A}$ be the closure of a set A. Then: (i) $\bar{A}$ is closed; (ii) if F is a closed superset of A, then $A \subset \bar{A} \subset F$; and (iii) A is closed iff $A = \bar{A}$.

Example 4.1: Consider the topology $\mathcal{T}$ on $X = \{a, b, c, d, e\}$ of Example 3.1 where the closed subsets of X are

$$\emptyset,\ X,\ \{b, c, d, e\},\ \{a, b, e\},\ \{b, e\},\ \{a\}$$

Accordingly, $\overline{\{b\}} = \{b, e\}$, $\overline{\{a, c\}} = X$, $\overline{\{b, d\}} = \{b, c, d, e\}$

Example 4.2: Let X be a cofinite topological space, i.e. the complements of finite sets and $\emptyset$ are the open sets. Then the closed sets are precisely the finite subsets of X together with X. Hence if $A \subset X$ is finite, its closure $\bar{A}$ is A itself since A is closed. On the other hand, if $A \subset X$ is infinite then X is the only closed superset of A; so $\bar{A}$ is X. More concisely, for any subset A of a cofinite space X,

$$\bar{A} = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{if } A \text{ is infinite} \end{cases}$$

The closure of a set can be completely described in terms of its limit points as follows:

Theorem 5.6: Let A be a subset of a topological space X. Then the closure of A is the union of A and its set of accumulation points, i.e.,

$$\bar{A} = A \cup A'$$

A point $p \in X$ is called a *closure point* or *adherent point* of $A \subset X$ iff p belongs to the closure of A, i.e. $p \in \bar{A}$. In view of the preceding theorem, $p \in X$ is a closure point of $A \subset X$ iff $p \in A$ or p is a limit point of A.

Example 4.3: Consider the set $\mathbf{Q}$ of rational numbers. As seen previously, in the usual topology for $\mathbf{R}$, every real number $a \in \mathbf{R}$ is a limit point of $\mathbf{Q}$. Hence the closure of $\mathbf{Q}$ is the entire set $\mathbf{R}$ of real numbers, i.e. $\bar{\mathbf{Q}} = \mathbf{R}$.

A subset A of a topological space X is said to be *dense* in $B \subset X$ if B is contained in the closure of A, i.e. $B \subset \bar{A}$. In particular, A is dense in X or is a dense subset of X iff $\bar{A} = X$.

Example 4.4: Observe in Example 4.1 that

$$\overline{\{a, c\}} = X \quad \text{and} \quad \overline{\{b, d\}} = \{b, c, d, e\}$$

where $X = \{a, b, c, d, e\}$. Hence the set $\{a, c\}$ is a dense subset of X but the set $\{b, d\}$ is not.

Example 4.5: As noted in Example 4.3, $\bar{\mathbf{Q}} = \mathbf{R}$. In other words, in the usual topology, the set $\mathbf{Q}$ of rational numbers is dense in $\mathbf{R}$.

The operator "closure", assigning to each subset A of X its closure $\bar{A} \subset X$ satisfies the four properties appearing in the proposition below, called the Kuratowski Closure Axioms. In fact, these axioms may be used to define a topology on X, as we shall prove subsequently.

Proposition 5.7: (i) $\bar{\emptyset} = \emptyset$; (ii) $A \subset \bar{A}$; (iii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$; and (iv) $(A^-)^- = \bar{A}$.

INTERIOR, EXTERIOR, BOUNDARY

Let A be a subset of a topological space X. A point $p \in A$ is called an *interior point* of A if p belongs to an open set G contained in A:

$$p \in G \subset A \quad \text{where } G \text{ is open}$$

The set of interior points of A, denoted by

$$\text{int}(A), \quad \overset{\circ}{A} \quad \text{or} \quad A^\circ$$

is called the *interior* of A. The interior of A can also be characterized as follows:

Proposition 5.8: The interior of a set A is the union of all open subsets of A. Furthermore: (i) A° is open; (ii) A° is the largest open subset of A, i.e. if G is an open subset of A then $G \subset A^\circ \subset A$; and (iii) A is open iff $A = A^\circ$.

The *exterior* of A, written ext(A), is the interior of the complement of A, i.e. int(A^c). The *boundary* of A, written b(A), is the set of points which do not belong to the interior or the exterior of A. Next follows an important relationship between interior, exterior and closure.

Theorem 5.9: Let A be any subset of a topological space X. Then the closure of A is the union of the interior and boundary of A, i.e. $\bar{A} = A^\circ \cup \text{b}(A)$.

Example 5.1: Consider the four intervals $[a,b]$, (a,b), $(a,b]$ and $[a,b)$ whose endpoints are a and b. The interior of each is the open interval (a,b) and the boundary of each is the set of endpoints, i.e. $\{a,b\}$.

Example 5.2: Consider the topology

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$$

on $X = \{a,b,c,d,e\}$, and the subset $A = \{b,c,d\}$ of X. The points c and d are each interior points of A since

$$c,d \in \{c,d\} \subset A$$

where $\{c,d\}$ is an open set. The point $b \in A$ is not an interior point of A; so $\text{int}(A) = \{c,d\}$. Only the point $a \in X$ is exterior to A, i.e. interior to the complement $A^c = \{a,e\}$ of A; hence $\text{int}(A^c) = \{a\}$. Accordingly the boundary of A consists of the points b and e, i.e. $\text{b}(A) = \{b,e\}$.

Example 5.3: Consider the set $\mathbf{Q}$ of rational numbers. Since every open subset of $\mathbf{R}$ contains both rational and irrational points, there are no interior or exterior points of $\mathbf{Q}$; so $\text{int}(\mathbf{Q}) = \emptyset$ and $\text{int}(\mathbf{Q}^c) = \emptyset$. Hence the boundary of $\mathbf{Q}$ is the entire set of real numbers, i.e. $\text{b}(\mathbf{Q}) = \mathbf{R}$.

A subset A of a topological space X is said to be *nowhere dense* in X if the interior of the closure of A is empty, i.e. $\text{int}(\bar{A}) = \emptyset$.

Example 5.4: Consider the subset $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ of $\mathbf{R}$. As noted previously, A has exactly one limit point, 0. Hence $\bar{A} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Observe that $\bar{A}$ has no interior points; so A is nowhere dense in $\mathbf{R}$.

Example 5.5: Let A consist of the rational points between 0 and 1, i.e. $A = \{x : x \in \mathbf{Q}, 0 < x < 1\}$. Observe that the interior of A is empty, i.e. $\text{int}(A) = \emptyset$. But A is not nowhere dense in $\mathbf{R}$; for the closure of A is $[0,1]$, and so

$$\text{int}(\bar{A}) = \text{int}([0,1]) = (0,1)$$

is not empty.

NEIGHBORHOODS AND NEIGHBORHOOD SYSTEMS

Let p be a point in a topological space X. A subset N of X is a *neighborhood* of p iff N is a superset of an open set G containing p:

$$p \in G \subset N \qquad \text{where } G \text{ is an open set}$$

In other words, the relation "N is a neighborhood of a point p" is the inverse of the relation "p is an interior point of N". The class of all neighborhoods of $p \in X$, denoted by $\mathcal{N}_p$, is called the *neighborhood system* of p.

Example 6.1: Let a be any real number, i.e. $a \in \mathbf{R}$. Then each closed interval $[a-\delta, a+\delta]$, with center a, is a neighborhood of a since it contains the open interval $(a-\delta, a+\delta)$ containing a. Similarly, if p is a point in the plane $\mathbf{R}^2$, then every closed disc $\{q \in \mathbf{R}^2 : d(p,q) \leq \delta \neq 0\}$, with center p, is a neighborhood of p since it contains the open disc with center p.

The central facts about the neighborhood system $\mathcal{N}_p$ of any point $p \in X$ are the four properties appearing in the proposition below, called the Neighborhood Axioms. In fact, these axioms may be used to define a topology on X, as we shall note subsequently.

Proposition 5.10: (i) $\mathcal{N}_p$ is not empty and p belongs to each member of $\mathcal{N}_p$.

(ii) The intersection of any two members of $\mathcal{N}_p$ belongs to $\mathcal{N}_p$.

(iii) Every superset of a member of $\mathcal{N}_p$ belongs to $\mathcal{N}_p$.

(iv) Each member $N \in \mathcal{N}_p$ is a superset of a member $G \in \mathcal{N}_p$ where G is a neighborhood of each of its points, i.e. $G \in \mathcal{N}_g$ for every $g \in G$.

CONVERGENT SEQUENCES

A sequence $\langle a_1, a_2, \ldots \rangle$ of points in a topological space X *converges* to a point $b \in X$, or b is the *limit* of the sequence $\langle a_n \rangle$, denoted by

$$\lim_{n\to\infty} a_n = b, \quad \lim a_n = b \quad \text{or} \quad a_n \to b$$

iff for each open set G containing b there exists a positive integer $n_0 \in N$ such that

$$n > n_0 \quad \text{implies} \quad a_n \in G$$

that is, if G contains almost all, i.e. all except a finite number, of the terms of the sequence.

Example 7.1: Let $\langle a_1, a_2, \ldots \rangle$ be a sequence of points in an indiscrete topological space $(X, \mathcal{J})$. Note that: (i) X is the only open set containing any point $b \in X$; and (ii) X contains every term of the sequence $\langle a_n \rangle$. Accordingly, the sequence $\langle a_1, a_2, \ldots \rangle$ converges to every point $b \in X$.

Example 7.2: Let $\langle a_1, a_2, \ldots \rangle$ be a sequence of points in a discrete topological space $(X, \mathcal{D})$. Now for every point $b \in X$, the singleton set $\{b\}$ is an open set containing b. So, if $a_n \to b$, then the set $\{b\}$ must contain almost all of the terms of the sequence. In other words, the sequence $\langle a_n \rangle$ converges to a point $b \in X$ iff the sequence is of the form $\langle a_1, a_2, \ldots, a_{n_0}, b, b, b, \ldots \rangle$.

Example 7.3: Let $\mathcal{T}$ be the topology on an infinite set X which consists of $\emptyset$ and the complements of countable sets (see Problem 56). We claim that a sequence $\langle a_1, a_2, \ldots \rangle$ in X converges to $b \in X$ iff the sequence is also of the form $\langle a_1, a_2, \ldots, a_{n_0}, b, b, b, \ldots \rangle$, i.e. the set A consisting of the terms of $\langle a_n \rangle$ different from b is finite. Now A is countable and so A^c is an open set containing b. Hence if $a_n \to b$ then A^c contains all except a finite number of the terms of the sequence, and so A is finite.

COARSER AND FINER TOPOLOGIES

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a non-empty set X. Suppose that each $\mathcal{T}_1$-open subset of X is also a $\mathcal{T}_2$-open subset of X. That is, suppose that $\mathcal{T}_1$ is a subclass of $\mathcal{T}_2$, i.e. $\mathcal{T}_1 \subset \mathcal{T}_2$. Then we say that $\mathcal{T}_1$ is *coarser* or *smaller* (sometimes called *weaker*) than $\mathcal{T}_2$ or that $\mathcal{T}_2$ is *finer* or *larger* than $\mathcal{T}_1$. Observe that the collection $\mathbf{T} = \{\mathcal{T}_i\}$ of all topologies on X is partially ordered by class inclusion; so we shall also write

$$\mathcal{T}_1 \lesssim \mathcal{T}_2 \quad \text{for} \quad \mathcal{T}_1 \subset \mathcal{T}_2$$

and we shall say that two topologies on X are *not comparable* if neither is coarser than the other.

Example 8.1: Consider the discrete topology $\mathcal{D}$, the indiscrete topology $\mathcal{J}$ and any other topology $\mathcal{T}$ on any set X. Then $\mathcal{T}$ is coarser than $\mathcal{D}$ and $\mathcal{T}$ is finer than $\mathcal{J}$. That is, $\mathcal{J} \lesssim \mathcal{T} \lesssim \mathcal{D}$.

Example 8.2: Consider the cofinite topology $\mathcal{T}$ and the usual topology $\mathcal{U}$ on the plane $\mathbf{R}^2$. Recall that every finite subset of $\mathbf{R}^2$ is a $\mathcal{U}$-closed set; hence the complement of any finite subset of $\mathbf{R}^2$, i.e. any member of $\mathcal{T}$, is also a $\mathcal{U}$-open set. In other words, $\mathcal{T}$ is coarser than $\mathcal{U}$, i.e. $\mathcal{T} \lesssim \mathcal{U}$.

SUBSPACES, RELATIVE TOPOLOGIES

Let A be a non-empty subset of a topological space $(X, \mathcal{T})$. The class $\mathcal{T}_A$ of all intersections of A with $\mathcal{T}$-open subsets of X is a topology on A; it is called the *relative topology* on A or the *relativization* of $\mathcal{T}$ to A, and the topological space $(A, \mathcal{T}_A)$ is called a *subspace* of $(X, \mathcal{T})$. In other words, a subset H of A is a $\mathcal{T}_A$-open set, i.e. open relative to A, if and only if there exists a $\mathcal{T}$-open subset G of X such that

$$H = G \cap A$$

Example 9.1: Consider the topology

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$$

on $X = \{a,b,c,d,e\}$, and the subset $A = \{a,d,e\}$ of X. Observe that

$$X \cap A = A, \quad \{a\} \cap A = \{a\}, \quad \{a,c,d\} \cap A = \{a,d\}$$
$$\emptyset \cap A = \emptyset, \quad \{c,d\} \cap A = \{d\}, \quad \{b,c,d,e\} \cap A = \{d,e\}$$

Hence the relativization of $\mathcal{T}$ to A is

$$\mathcal{T}_A = \{A, \emptyset, \{a\}, \{d\}, \{a,d\}, \{d,e\}\}$$

Example 9.2: Consider the usual topology $\mathcal{U}$ on $\mathbf{R}$ and the relative topology $\mathcal{T}_A$ on the closed interval $A = [3,8]$. Note that the closed-open interval $[3,5)$ is open in the relative topology on A, i.e. is a $\mathcal{T}_A$-open set, since

$$[3,5) = (2,5) \cap A$$

where $(2,5)$ is a $\mathcal{T}$-open subset of $\mathbf{R}$. Thus we see that a set may be open relative to a subspace but be neither open nor closed in the entire space.

EQUIVALENT DEFINITIONS OF TOPOLOGIES

Our definition of a topological space gave axioms for the open sets in the topological space, that is, we used the open set as the primitive notion for the topology. We now state two theorems which exhibit alternate methods of defining a topology on a set, using as primitives the notions of "neighborhood of a point" and "closure of a set".

Theorem 5.11: Let X be a non-empty set and let there be assigned to each point $p \in X$ a class $\mathcal{A}_p$ of subsets of X satisfying the following axioms:

[A₁] $\mathcal{A}_p$ is not empty and p belongs to each member of $\mathcal{A}_p$.

[A₂] The intersection of any two members of $\mathcal{A}_p$ belongs to $\mathcal{A}_p$.

[A₃] Every superset of a member of $\mathcal{A}_p$ belongs to $\mathcal{A}_p$.

[A₄] Each member $N \in \mathcal{A}_p$ is a superset of a member $G \in \mathcal{A}_p$ such that $G \in \mathcal{A}_g$ for every $g \in G$.

Then there exists one and only one topology $\mathcal{T}$ on X such that $\mathcal{A}_p$ is the $\mathcal{T}$-neighborhood system of the point $p \in X$.

Theorem 5.12: Let X be a non-empty set and let k be an operation which assigns to each subset A of X the subset A^k of X, satisfying the following axioms, called the Kuratowski Closure Axioms:

[K₁] $\emptyset^k = \emptyset$

[K₂] $A \subset A^k$

[K₃] $(A \cup B)^k = A^k \cup B^k$

[K₄] $(A^k)^k = A^k$

Then there exists one and only one topology $\mathcal{T}$ on X such that A^k will be the $\mathcal{T}$-closure of the subset A of X.

Solved Problems

TOPOLOGIES, OPEN SETS

1. Let $X = \{a, b, c, d, e\}$. Determine whether or not each of the following classes of subsets of X is a topology on X.

(i) $\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$

(ii) $\mathcal{T}_2 = \{X, \emptyset, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$

(iii) $\mathcal{T}_3 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}\}$

Solution:

(i) $\mathcal{T}_1$ is not a topology on X since

$$\{a, b\}, \{a, c\} \in \mathcal{T}_1 \quad \text{but} \quad \{a, b\} \cup \{a, c\} = \{a, b, c\} \notin \mathcal{T}_1$$

(ii) $\mathcal{T}_2$ is not a topology on X since

$$\{a, b, c\}, \{a, b, d\} \in \mathcal{T}_2 \quad \text{but} \quad \{a, b, c\} \cap \{a, b, d\} = \{a, b\} \notin \mathcal{T}_2$$

(iii) $\mathcal{T}_3$ is a topology on X since it satisfies the necessary axioms.

2. Let $\mathcal{T}$ be the class consisting of $\mathbf{R}$, $\emptyset$ and all infinite open intervals $A_q = (q, \infty)$ with $q \in \mathbf{Q}$, the rationals. Show that $\mathcal{T}$ is not a topology on $\mathbf{R}$.

Solution:

Observe that $$A = \cup \{A_q : q \in \mathbf{Q}, q > \sqrt{2}\} = (\sqrt{2}, \infty)$$

is the union of members of $\mathcal{T}$, but $A \notin \mathcal{T}$ since $\sqrt{2}$ is irrational. Hence $\mathcal{T}$ violates $[\mathbf{O}_2]$ and is therefore not a topology on $\mathbf{R}$.

3. Let $\mathcal{T}$ be a topology on a set X consisting of four sets, i.e.

$$\mathcal{T} = \{X, \emptyset, A, B\}$$

where A and B are non-empty distinct proper subsets of X. What conditions must A and B satisfy?

Solution:

Since $A \cap B$ must also belong to $\mathcal{T}$, there are two possibilities:

Case I. $A \cap B = \emptyset$

Then $A \cup B$ cannot be A or B; hence $A \cup B = X$. Thus the class $\{A, B\}$ is a partition of X.

Case II. $A \cap B = A$ or $A \cap B = B$

In either case, one of the sets is a subset of the other, and the members of $\mathcal{T}$ are totally ordered by inclusion: $\emptyset \subset A \subset B \subset X$ or $\emptyset \subset B \subset A \subset X$.

4. List all topologies on $X = \{a, b, c\}$ which consist of exactly four members.

Solution:

Each topology $\mathcal{T}$ on X with four members is of the form $\mathcal{T} = \{X, \emptyset, A, B\}$ where A and B correspond to Case I or Case II of the preceding problem.

Case I. $\{A, B\}$ is a partition of X.

The topologies in this case are the following:

$$\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{b, c\}\}, \quad \mathcal{T}_2 = \{X, \emptyset, \{b\}, \{a, c\}\}, \quad \mathcal{T}_3 = \{X, \emptyset, \{c\}, \{a, b\}\}$$

Case II. The members of $\mathcal{T}$ are totally ordered by inclusion.

The topologies in this case are the following:

$$\mathcal{T}_4 = \{X, \emptyset, \{a\}, \{a, b\}\} \qquad \mathcal{T}_7 = \{X, \emptyset, \{b\}, \{a, b\}\}$$
$$\mathcal{T}_5 = \{X, \emptyset, \{a\}, \{a, c\}\} \qquad \mathcal{T}_8 = \{X, \emptyset, \{c\}, \{a, c\}\}$$
$$\mathcal{T}_6 = \{X, \emptyset, \{b\}, \{b, c\}\} \qquad \mathcal{T}_9 = \{X, \emptyset, \{c\}, \{b, c\}\}$$

5. Let $f: X \to Y$ be a function from a non-empty set X into a topological space $(Y, \mathcal{U})$. Furthermore, let $\mathcal{T}$ be the class of inverses of open subsets of Y:
$$\mathcal{T} = \{f^{-1}[G] : G \in \mathcal{U}\}$$
Show that $\mathcal{T}$ is a topology on X.

Solution:

Since $\mathcal{U}$ is a topology, $Y, \emptyset \in \mathcal{U}$. But $X = f^{-1}[Y]$ and $\emptyset = f^{-1}[\emptyset]$, so $X, \emptyset \in \mathcal{T}$ and $\mathcal{T}$ satisfies **[O₁]**.

Let $\{A_i\}$ be a class of sets in $\mathcal{T}$. By definition, there exist $G_i \in \mathcal{U}$ for which $A_i = f^{-1}[G_i]$. But
$$\cup_i A_i = \cup_i f^{-1}[G_i] = f^{-1}[\cup_i G_i]$$
Since $\mathcal{U}$ is a topology, $\cup_i G_i \in \mathcal{U}$, so $\cup_i A_i \in \mathcal{T}$, and $\mathcal{T}$ satisfies **[O₂]**.

Lastly, let $A_1, A_2 \in \mathcal{T}$. Then
$$\exists\, G_1, G_2 \in \mathcal{U} \quad \text{such that} \quad A_1 = f^{-1}[G_1],\ A_2 = f^{-1}[G_2]$$
But
$$A_1 \cap A_2 = f^{-1}[G_1] \cap f^{-1}[G_2] = f^{-1}[G_1 \cap G_2]$$
and $G_1 \cap G_2 \in \mathcal{U}$. Thus $A_1 \cap A_2 \in \mathcal{T}$ and **[O₃]** is also satisfied.

6. Consider the second axiom for a topology $\mathcal{T}$ on a set X:

[O₂] The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

Show that **[O₂]** can be replaced by the following weaker axiom:

[O′₂] The union of any number of sets in $\mathcal{T} \setminus \{X, \emptyset\}$ belongs to $\mathcal{T}$.

In other words, show that the axioms **[O₁]**, **[O′₂]** and **[O₃]** are equivalent to the axioms **[O₁]**, **[O₂]** and **[O₃]**.

Solution:

Let $\mathcal{T}$ be a class of subsets of X satisfying **[O₁]**, **[O′₂]** and **[O₃]**, and let $\mathcal{A}$ be a subclass of $\mathcal{T}$. We want to show that $\mathcal{T}$ also satisfies **[O₂]**, i.e. that $\bigcup\{E : E \in \mathcal{A}\} \in \mathcal{T}$.

Case I. $X \in \mathcal{A}$.

Then $\bigcup\{E : E \in \mathcal{A}\} = X$ and therefore belongs to $\mathcal{T}$ by **[O₁]**.

Case II. $X \notin \mathcal{A}$.

Then
$$\bigcup\{E : E \in \mathcal{A}\} = \bigcup\{E : E \in \mathcal{A} \setminus \{X\}\}$$
But the empty set $\emptyset$ does not contribute any elements to a union of sets; hence
$$\bigcup\{E : E \in \mathcal{A}\} = \bigcup\{E : E \in \mathcal{A} \setminus \{X\}\} = \bigcup\{E : E \in \mathcal{A} \setminus \{X, \emptyset\}\} \qquad (1)$$
Since $\mathcal{A}$ is a subclass of $\mathcal{T}$, $\mathcal{A} \setminus \{X, \emptyset\}$ is a subclass of $\mathcal{T} \setminus \{X, \emptyset\}$, so by **[O′₂]** the union in *(1)* belongs to $\mathcal{T}$.

7. Prove: Let A be a subset of a topological space X with the property that each point $p \in A$ belongs to an open set G_p contained in A. Then A is open.

Solution:

For each point $p \in A$, $p \in G_p \subset A$. Hence $\bigcup\{G_p : p \in A\} = A$ and so A is a union of open sets and, by **[O₂]**, is open.

8. Let $\mathcal{T}$ be a class of subsets of X totally ordered by set inclusion. Show that $\mathcal{T}$ satisfies **[O₃]**, i.e. the intersection of any two members of $\mathcal{T}$ belongs to $\mathcal{T}$.

Solution:

Let $A, B \in \mathcal{T}$. Since $\mathcal{T}$ is totally ordered by set inclusion,
$$\text{either} \quad A \cap B = A \quad \text{or} \quad A \cap B = B$$
In either case $A \cap B \in \mathcal{T}$, and so $\mathcal{T}$ satisfies **[O₃]**.

9. Let $\mathcal{T}$ be the class of subsets of **R** consisting of **R**, $\emptyset$ and all open infinite $E_a = (a, \infty)$ with $a \in \mathbf{R}$. Show that $\mathcal{T}$ is a topology on **R**.

Solution:

Since **R** and $\emptyset$ belong to $\mathcal{T}$, $\mathcal{T}$ satisfies $[\mathbf{O}_1]$. Observe that $\mathcal{T}$ is totally ordered by set inclusion; hence $\mathcal{T}$ satisfies $[\mathbf{O}_3]$.

Now let $\mathcal{A}$ be a subclass of $\mathcal{T} \setminus \{X, \emptyset\}$, that is $\mathcal{A} = \{E_i : i \in I\}$ where I is some set of real numbers. We want to show that $\cup_i E_i$ belongs to $\mathcal{T}$. If I is not bounded from below, i.e. if $\inf(I) = -\infty$, then $\cup_i E_i = \mathbf{R}$. If I is bounded from below, say $\inf(I) = i_0$, then $\cup_i E_i = (i_0, \infty) = E_{i_0}$. In either case, $\cup_i E_i \in \mathcal{T}$, and $\mathcal{T}$ satisfies $[\mathbf{O}_2']$.

10. Let $\mathcal{T}$ be the class of subsets of **N** consisting of $\emptyset$ and all subsets of **N** of the form $E_n = \{n, n+1, n+2, \ldots\}$ with $n \in \mathbf{N}$.

(i) Show that $\mathcal{T}$ is a topology on **N**.

(ii) List the open sets containing the positive integer 6.

Solution:

(i) Since $\emptyset$ and $E_1 = \{1, 2, 3, \ldots\} = \mathbf{N}$ belong to $\mathcal{T}$, $\mathcal{T}$ satisfies $[\mathbf{O}_1]$. Furthermore, since $\mathcal{T}$ is totally ordered by set inclusion, $\mathcal{T}$ also satisfies $[\mathbf{O}_3]$.

Now let $\mathcal{A}$ be a subclass of $\mathcal{T} \setminus \{\mathbf{N}, \emptyset\}$, that is, $\mathcal{A} = \{E_n : n \in I\}$ where I is some set of positive integers. Note that I contains a smallest positive integer n_0 and

$$\cup \{E_n : n \in I\} = \{n_0, n_0 + 1, n_0 + 2, \ldots\} = E_{n_0}$$

which belongs to $\mathcal{T}$. Hence $\mathcal{T}$ satisfies $[\mathbf{O}_2']$, and so $\mathcal{T}$ is a topology on **N**.

(ii) Since the non-empty open sets are of the form

$$E_n = \{n, n+1, n+2, \ldots\}$$

with $n \in \mathbf{N}$, the open sets containing the positive integer 6 are the following:

$$E_1 = \mathbf{N} = \{1, 2, 3, \ldots\} \qquad E_4 = \{4, 5, 6, \ldots\}$$
$$E_2 = \{2, 3, 4, \ldots\} \qquad E_5 = \{5, 6, 7, \ldots\}$$
$$E_3 = \{3, 4, 5, \ldots\} \qquad E_6 = \{6, 7, 8, \ldots\}$$

ACCUMULATION POINTS, DERIVED SETS

11. Let $\mathcal{T}$ be the topology on **N** which consists of $\emptyset$ and all subsets of **N** of the form $E_n = \{n, n+1, n+2, \ldots\}$ where $n \in \mathbf{N}$ as in Problem 10.

(i) Find the accumulation points of the set $A = \{4, 13, 28, 37\}$.

(ii) Determine those subsets E of **N** for which $E' = \mathbf{N}$.

Solution:

(i) Observe that the open sets containing any point $p \in \mathbf{N}$ are the sets E_i where $i \leq p$. If $n_0 \leq 36$, then every open set containing n_0 also contains $37 \in A$ which is different from n_0; hence $n_0 \leq 36$ is a limit point of A. On the other hand, if $n_0 > 36$ then the open set $E_{n_0} = \{n_0, n_0 + 1, n_0 + 2, \ldots\}$ contains no point of A different from n_0. So $n_0 > 36$ is not a limit point of A. Accordingly, the derived set of A is $A' = \{1, 2, 3, \ldots, 34, 35, 36\}$.

(ii) If E is an infinite subset of **N** then E is not bounded from above. So every open set containing any point $p \in \mathbf{N}$ will contain points of E other than p. Hence $E' = \mathbf{N}$.

On the other hand, if E is finite then E is bounded from above, say, by $n_0 \in \mathbf{N}$. Then the open set E_{n_0+1} contains no point of E. Hence $n_0 + 1 \in \mathbf{N}$ is not a limit point of E, and so $E' \neq \mathbf{N}$.

12. Let A be a subset of a topological space $(X, \mathcal{T})$. When will a point $p \in X$ not be a limit point of A?

Solution:

The point $p \in X$ is a limit point of A iff every open neighborhood of p contains a point of A other than p, i.e.,

$$p \in G \text{ and } G \in \mathcal{T} \quad \text{implies} \quad (G \setminus \{p\}) \cap A \neq \emptyset$$

So p is not a limit point of A if there exists an open set G such that

$$p \in G \quad \text{and} \quad (G \setminus \{p\}) \cap A = \emptyset$$

or, equivalently, $$p \in G \quad \text{and} \quad G \cap A = \emptyset \text{ or } G \cap A = \{p\}$$

or, equivalently, $$p \in G \quad \text{and} \quad G \cap A \subset \{p\}$$

13. Let A be any subset of a discrete topological space X. Show that the derived set A' of A is empty.

Solution:

Let p be any point in X. Recall that every subset of a discrete space is open. Hence, in particular, the singleton set $G = \{p\}$ is an open subset of X. But

$$p \in G \quad \text{and} \quad G \cap A = (\{p\} \cap A) \subset \{p\}$$

Hence, by the above problem, $p \notin A'$ for every $p \in X$, i.e. $A' = \emptyset$.

14. Consider the topology

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}\}$$

on $X = \{a,b,c,d,e\}$. Determine the derived sets of (i) $A = \{c,d,e\}$ and (ii) $B = \{b\}$.

Solution:

(i) Note that $G = \{a\}$ and $H = \{a,b,e\}$ are open subsets of X and that

$$a \in G = \{a\} \quad \text{and} \quad G \cap A = \emptyset$$

$$e \in H = \{a,b,e\} \quad \text{and} \quad H \cap A = \{e\}$$

Hence a and e are not limit points of A. On the other hand, every other point in X is a limit point of A since every open set containing it also contains a point of A different from it. Thus $A' = \{b,c,d\}$.

(ii) Note that $\{a\}$, $\{a,b\}$ and $\{a,c,d\}$ are open subsets of X and that

$$a \in \{a\} \quad \text{and} \quad \{a\} \cap B = \emptyset$$

$$b \in \{a,b\} \quad \text{and} \quad \{a,b\} \cap B = \{b\}$$

$$c,d \in \{a,c,d\} \quad \text{and} \quad \{a,c,d\} \cap B = \emptyset$$

Hence a, b, c and d are not limit points of $B = \{b\}$. But e is a limit point of B since the open sets containing e are $\{a,b,e\}$ and X and each contains the point $b \in B$ different from e. Thus $B' = \{e\}$.

15. Prove: If A is a subset of B, then every limit point of A is also a limit point of B, i.e., $A \subset B$ implies $A' \subset B'$.

Solution:

Recall that $p \in A'$ iff $(G \setminus \{p\}) \cap A \neq \emptyset$ for every open set G containing p. But $B \supset A$; hence

$$(G \setminus \{p\}) \cap B \supset (G \setminus \{p\}) \cap A \neq \emptyset$$

So $p \in A'$ implies $p \in B'$, i.e. $A' \subset B'$.

16. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X such that $\mathcal{T}_1 \subset \mathcal{T}_2$, i.e. every $\mathcal{T}_1$-open subset of X is also a $\mathcal{T}_2$-open subset of X. Furthermore, let A be any subset of X.

(i) Show that every $\mathcal{T}_2$-limit point of A is also a $\mathcal{T}_1$-limit point of A.

(ii) Construct a space in which a $\mathcal{T}_1$-limit point is not a $\mathcal{T}_2$-limit point.

Solution:

(i) Let p be a $\mathcal{T}_2$-limit point of A; i.e. $(G \setminus \{p\}) \cap A \neq \emptyset$ for every $G \in \mathcal{T}_2$ such that $p \in G$. But $\mathcal{T}_1 \subset \mathcal{T}_2$; so, in particular, $(G \setminus \{p\}) \cap A \neq \emptyset$ for every $G \in \mathcal{T}_1$ such that $p \in G$, i.e. p is a $\mathcal{T}_1$-limit point of A.

(ii) Consider the usual topology $\mathcal{U}$ and the discrete topology $\mathcal{D}$ on $\mathbf{R}$. Note that $\mathcal{U} \subset \mathcal{D}$ since $\mathcal{D}$ contains every subset of $\mathbf{R}$. By Problem 13, 0 is not a $\mathcal{D}$-limit point of the set $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ since A' is empty. But 0 is a limit point of A with respect to the usual topology on $\mathbf{R}$.

17. Prove: Let A and B be subsets of a topological space $(X, \mathcal{T})$. Then $(A \cup B)' = A' \cup B'$.

Solution:

Utilizing Problem 15, $A \subset A \cup B$ implies $A' \subset (A \cup B)'$

$B \subset A \cup B$ implies $B' \subset (A \cup B)'$

So $A' \cup B' \subset (A \cup B)'$, and we need only show that

$$(A \cup B)' \subset A' \cup B'$$

Assume $p \notin A' \cup B'$; thus $\exists G, H \in \mathcal{T}$ such that

$$p \in G \text{ and } G \cap A \subset \{p\} \quad \text{and} \quad p \in H \text{ and } H \cap B \subset \{p\}$$

But $G \cap H \in \mathcal{T}$, $p \in G \cap H$ and

$$(G \cap H) \cap (A \cup B) = (G \cap H \cap A) \cup (G \cap H \cap B) \subset (G \cap A) \cup (H \cap B) \subset \{p\} \cup \{p\} = \{p\}$$

Thus $p \notin (A \cup B)'$, and so $(A \cup B)' \subset (A' \cup B')$.

CLOSED SETS, CLOSURE OPERATION, DENSE SETS

18. Consider the following topology on $X = \{a, b, c, d, e\}$:

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$$

(i) List the closed subsets of X.

(ii) Determine the closure of the sets $\{a\}$, $\{b\}$ and $\{c, e\}$.

(iii) Which sets in (ii) are dense in X?

Solution:

(i) A set is closed iff its complement is open. Hence write the complement of each set in $\mathcal{T}$:

$$\emptyset,\ X,\ \{b, c, d, e\},\ \{c, d, e\},\ \{b, e\},\ \{e\},\ \{c, d\}$$

(ii) The closure $\bar{A}$ of any set A is the intersection of all closed supersets of A. The only closed superset of $\{a\}$ is X; the closed supersets of $\{b\}$ are $\{b, e\}$, $\{b, c, d, e\}$ and X; and the closed supersets of $\{c, e\}$ are $\{c, d, e\}$, $\{b, c, d, e\}$ and X. Thus,

$$\overline{\{a\}} = X, \quad \overline{\{b\}} = \{b, e\}, \quad \overline{\{c, e\}} = \{c, d, e\}$$

(iii) A set A is dense in X iff $\bar{A} = X$; so $\{a\}$ is the only dense set.

19. Let $\mathcal{T}$ be the topology on $\mathbf{N}$ which consists of $\emptyset$ and all subsets of $\mathbf{N}$ of the form $E_n = \{n, n+1, n+2, \ldots\}$ where $n \in \mathbf{N}$ as in Problem 10.

(i) Determine the closed subsets of $(\mathbf{N}, \mathcal{T})$.

(ii) Determine the closure of the sets $\{7, 24, 47, 85\}$ and $\{3, 6, 9, 12, \ldots\}$.

(iii) Determine those subsets of $\mathbf{N}$ which are dense in $\mathbf{N}$.

Solution:

(i) A set is closed iff its complement is open. Hence the closed subsets of $\mathbf{N}$ are as follows:

$$\mathbf{N},\ \emptyset,\ \{1\},\ \{1, 2\},\ \{1, 2, 3\},\ \ldots,\ \{1, 2, \ldots, m\},\ \ldots$$

(ii) The closure of a set is the smallest closed superset. So

$$\overline{\{7,24,47,85\}} = \{1,2,\ldots,84,85\}, \quad \overline{\{3,6,9,12,\ldots\}} = \{1,2,3,\ldots\} = \mathbf{N}$$

(iii) If a subset A of $\mathbf{N}$ is infinite, or equivalently unbounded, then $\bar{A} = \mathbf{N}$, i.e. A is dense in $\mathbf{N}$. If A is finite then its closure is not $\mathbf{N}$, i.e. A is not dense in $\mathbf{N}$.

20. Let $\mathcal{T}$ be the topology on $\mathbf{R}$ consisting of $\mathbf{R}$, $\emptyset$ and all open infinite intervals $E_a = (a, \infty)$ where $a \in \mathbf{R}$.

(i) Determine the closed subsets of $(\mathbf{R}, \mathcal{T})$.

(ii) Determine the closure of the sets $[3,7)$, $\{7,24,47,85\}$ and $\{3,6,9,12,\ldots\}$.

Solution:

(i) A set is closed iff its complement is open. Hence the closed subsets of $(\mathbf{R}, \mathcal{T})$ are $\emptyset$, $\mathbf{R}$ and all closed infinite intervals $E_a^c = (-\infty, a]$.

(ii) The closure of a set is the smallest closed superset. Hence

$$\overline{[3,7)} = (-\infty, 7], \quad \overline{\{7,24,47,85\}} = (-\infty, 85], \quad \overline{\{3,6,9,12,\ldots\}} = (-\infty, \infty) = \mathbf{R}$$

21. Let X be a discrete topological space. (i) Determine the closure of any subset A of X. (ii) Determine the dense subsets of X.

Solution:

(i) Recall that in a discrete space X any $A \subset X$ is closed; hence $\bar{A} = A$.

(ii) A is dense in X iff $\bar{A} = X$. But $\bar{A} = A$, so X is the only dense subset of X.

22. Let X be an indiscrete space. (i) Determine the closed subsets of X. (ii) Determine the closure of any subset A of X. (iii) Determine the dense subsets of X.

Solution:

(i) Recall that the only open subsets of an indiscrete space X are X and $\emptyset$; hence the closed subsets of X are also X and $\emptyset$.

(ii) If $A = \emptyset$, then $\bar{A} = \emptyset$. If $A \neq \emptyset$, then X is the only closed superset of A; so $\bar{A} = X$. That is, for any $A \subset X$,

$$\bar{A} = \begin{cases} \emptyset & \text{if } A = \emptyset \\ X & \text{if } A \neq \emptyset \end{cases}$$

(iii) $A \subset X$ is dense in X iff $\bar{A} = X$; hence every non-empty subset of X is dense in X.

23. Prove Theorem 5.4: A subset A of a topological space X is closed if and only if A contains each of its accumulation points, i.e. $A' \subset A$.

Solution:

Suppose A is closed, and let $p \notin A$, i.e. $p \in A^c$. But A^c, the complement of a closed set, is open; hence $p \notin A'$ for A^c is an open set such that

$$p \in A^c \quad \text{and} \quad A^c \cap A = \emptyset$$

Thus $A' \subset A$ if A is closed.

Now assume $A' \subset A$; we show that A^c is open. Let $p \in A^c$; then $p \notin A'$, so $\exists$ an open set G such that

$$p \in G \quad \text{and} \quad (G \setminus \{p\}) \cap A = \emptyset$$

But $p \notin A$; hence

$$G \cap A = (G \setminus \{p\}) \cap A = \emptyset$$

So $G \subset A^c$. Thus p is an interior point of A^c, and so A^c is open.

24. Prove: If F is a closed superset of any set A, then $A' \subset F$.

Solution:

By Problem 15, $A \subset F$ implies $A' \subset F'$. But $F' \subset F$, by Theorem 5.4, since F is closed. Thus $A' \subset F' \subset F$, which implies $A' \subset F$.

25. Prove: $A \cup A'$ is a closed set.

Solution:

Let $p \in (A \cup A')^c$. Since $p \notin A'$, $\exists$ an open set G such that

$$p \in G \quad \text{and} \quad G \cap A = \emptyset \text{ or } \{p\}$$

However, $p \notin A$; hence, in particular, $G \cap A = \emptyset$.

We also claim that $G \cap A' = \emptyset$. For if $g \in G$, then

$$g \in G \quad \text{and} \quad G \cap A = \emptyset$$

where G is an open set. So $g \notin A'$ and thus $G \cap A' = \emptyset$. Accordingly,

$$G \cap (A \cup A') = (G \cap A) \cup (G \cap A') = \emptyset \cup \emptyset = \emptyset$$

and so $G \subset (A \cup A')^c$. Thus p is an interior point of $(A \cup A')^c$ which is therefore an open set. Hence $A \cup A'$ is closed.

26. Prove Theorem 5.6: $\bar{A} = A \cup A'$.

Solution:

Since $A \subset \bar{A}$ and $\bar{A}$ is closed, $A' \subset (\bar{A})' \subset \bar{A}$ and hence $A \cup A' \subset \bar{A}$. But $A \cup A'$ is a closed set containing A, so $A \subset \bar{A} \subset A \cup A'$. Thus $\bar{A} = A \cup A'$.

27. Prove: If $A \subset B$ then $\bar{A} \subset \bar{B}$.

Solution:

If $A \subset B$, then by Problem 15, $A' \subset B'$. So $A \cup A' \subset B \cup B'$ or, by the preceding problem, $\bar{A} \subset \bar{B}$.

28. Prove: $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Solution:

Utilizing the preceding problem, $\bar{A} \subset \overline{A \cup B}$ and $\bar{B} \subset \overline{A \cup B}$; hence $(\bar{A} \cup \bar{B}) \subset \overline{A \cup B}$. But $(A \cup B) \subset (\bar{A} \cup \bar{B})$, a closed set since it is the union of two closed sets. Then (Proposition 5.5) $(A \cup B) \subset \overline{A \cup B} \subset (\bar{A} \cup \bar{B})$ and therefore $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

29. Prove Proposition 5.7: (i) $\bar{\emptyset} = \emptyset$; (ii) $A \subset \bar{A}$; (iii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$; and (iv) $(A^-)^- = A^-$.

Solution:

(i) and (iv): $\emptyset$ and $\bar{A}$ are closed; hence they are equal to their closures. (ii) $A \subset A \cup A' = \bar{A}$ (Problem 26). (iii) Preceding problem.

INTERIOR, EXTERIOR, BOUNDARY

30. Consider the following topology on $X = \{a, b, c, d, e\}$:

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$$

(i) Find the interior points of the subset $A = \{a, b, c\}$ of X. (ii) Find the exterior points of A. (iii) Find the boundary points of A.

Solution:

(i) The points a and b are interior points of A since

$$a, b \in \{a, b\} \subset A = \{a, b, c\}$$

where $\{a, b\}$ is an open set, i.e. since each belongs to an open set contained in A. Note that c is not an interior point of A since c does not belong to any open set contained in A. Hence $\text{int}(A) = \{a, b\}$ is the interior of A.

(ii) The complement of A is $A^c = \{d, e\}$. Neither d nor e are interior points of A^c since neither belongs to any open subset of $A^c = \{d, e\}$. Hence $\text{int}(A^c) = \emptyset$, i.e. there are no exterior points of A.

(iii) The boundary $\text{b}(A)$ of A consists of those points which are neither interior nor exterior to A. So $\text{b}(A) = \{c, d, e\}$.

31. Prove Proposition 5.8: The interior of a set A is the union of all open subsets of A. Furthermore: (i) A° is open; (ii) A° is the largest open subset of A, i.e. if G is an open subset of A then $G \subset A^\circ \subset A$; and (iii) A is open iff $A = A^\circ$.

Solution:

Let $\{G_i\}$ be the class of all open subsets of A. If $x \in A^\circ$, then x belongs to an open subset of A, i.e.,

$$\exists i_0 \quad \text{such that} \quad x \in G_{i_0}$$

Hence $x \in \cup_i G_i$ and so $A^\circ \subset \cup_i G_i$. On the other hand, if $y \in \cup_i G_i$, then $y \in G_{i_0}$ for some i_0. Thus $y \in A^\circ$, and $\cup_i G_i \subset A^\circ$. Accordingly, $A^\circ = \cup_i G_i$.

(i) $A^\circ = \cup_i G_i$ is open since it is the union of open sets.

(ii) If G is an open subset of A then $G \in \{G_i\}$; so $G \subset \cup_i G_i = A^\circ \subset A$.

(iii) If A is open then $A \subset A^\circ \subset A$ or $A = A^\circ$. If $A = A^\circ$ then A is open since A° is open.

32. Let A be a non-empty proper subset of an indiscrete space X. Find the interior, exterior and boundary of A.

Solution:

X and $\emptyset$ are the only open subsets of X. Since $X \neq A$, $\emptyset$ is the only open subset of A; hence $\text{int}(A) = \emptyset$. Similarly, $\text{int}(A^c) = \emptyset$, i.e. the exterior of A is empty. Thus $\text{b}(A) = X$.

33. Let $\mathcal{T}$ be the topology on $\mathbf{R}$ consisting of $\mathbf{R}$, $\emptyset$ and all open infinite intervals $E_a = (a, \infty)$ where $a \in \mathbf{R}$. Find the interior, exterior and boundary of the closed infinite interval $A = [7, \infty)$.

Solution:

Since the interior of A is the largest open subset of A, $\text{int}(A) = (7, \infty)$. Note that $A^c = (-\infty, 7)$ contains no open set except $\emptyset$; so $\text{int}(A^c) = \text{ext}(A) = \emptyset$. The boundary consists of those points which do not belong to $\text{int}(A)$ or $\text{ext}(A)$; hence $\text{b}(A) = (-\infty, 7]$.

34. Prove Theorem 5.9: $\bar{A} = \text{int}(A) \cup \text{b}(A)$

Solution:

Since $X = \text{int}(A) \cup \text{b}(A) \cup \text{ext}(A)$, $(\text{int}(A) \cup \text{b}(A))^c = \text{ext}(A)$ and it suffices to show $(\bar{A})^c = \text{ext}(A)$.

Let $p \in \text{ext}(A)$; then $\exists$ an open G such that

$$p \in G \subset A^c \quad \text{which implies} \quad G \cap A = \emptyset$$

So p is not a limit point of A, i.e. $p \notin A'$, and $p \notin A$. Hence $p \notin A' \cup A = \bar{A}$. In other words, $\text{ext}(A) \subset (\bar{A})^c$.

Now assume $p \in (\bar{A})^c = (A \cup A')^c$. Thus $p \notin A'$, so $\exists$ an open set G such that

$$p \in G \quad \text{and} \quad (G \setminus \{p\}) \cap A = \emptyset$$

But also $p \notin A$, so $G \cap A = \emptyset$ and $p \in G \subset A^c$. Thus $p \in \text{ext}(A)$, and $(\bar{A})^c = \text{ext}(A)$.

35. Show by a counterexample that the function f which assigns to each set its interior, i.e. $f(A) = \text{int}(A)$, does not commute with the function g which assigns to each set its closure, i.e. $g(A) = \bar{A}$.

Solution:

Consider $\mathbf{Q}$, the set of rational numbers, as a subset of $\mathbf{R}$ with the usual topology. Recall (Example 5.3) that the interior of $\mathbf{Q}$ is empty; hence

$$(g \circ f)(\mathbf{Q}) = g(f(\mathbf{Q})) = g(\text{int}(\mathbf{Q})) = g(\emptyset) = \bar{\emptyset} = \emptyset$$

On the other hand, $\bar{\mathbf{Q}} = \mathbf{R}$ and the interior of $\mathbf{R}$ is $\mathbf{R}$ itself. So

$$(f \circ g)(\mathbf{Q}) = f(g(\mathbf{Q})) = f(\bar{\mathbf{Q}}) = f(\mathbf{R}) = \mathbf{R}$$

Thus $g \circ f \neq f \circ g$, or f and g do not commute.

NEIGHBORHOODS, NEIGHBORHOOD SYSTEMS

36. Consider the following topology on $X = \{a, b, c, d, e\}$:

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$$

List the neighborhoods (i) of the point e, (ii) of the point c.

Solution:

(i) A neighborhood of e is any superset of an open set containing e. The open sets containing e are $\{a, b, e\}$ and X. The supersets of $\{a, b, e\}$ are $\{a, b, e\}$, $\{a, b, c, e\}$, $\{a, b, d, e\}$ and X; the only superset of X is X. Accordingly, the class of neighborhoods of e, i.e. the neighborhood system of e, is

$$\mathcal{N}_e = \{\{a, b, e\}, \{a, b, c, e\}, \{a, b, d, e\}, X\}$$

(ii) The open sets containing c are $\{a, c, d\}$, $\{a, b, c, d\}$ and X. Hence the neighborhood system of c is

$$\mathcal{N}_c = \{\{a, c, d\}, \{a, b, c, d\}, \{a, c, d, e\}, X\}$$

37. Determine the neighborhood system of a point p in an indiscrete space X.

Solution:

X and $\emptyset$ are the only open subsets of X; hence X is the only open set containing p. In addition, X is the only superset of X. Hence $\mathcal{N}_p = \{X\}$.

38. Prove: The intersection $N \cap M$ of any two neighborhoods N and M of a point p is also a neighborhood of p.

Solution:

N and M are neighborhoods of p, so $\exists$ open sets G, H such that

$$p \in G \subset N \quad \text{and} \quad p \in H \subset M$$

Hence $p \in G \cap H \subset N \cap M$, and $G \cap H$ is open, or $N \cap M$ is a neighborhood of p.

39. Prove: Any superset M of a neighborhood N of a point p is also a neighborhood of p.

Solution:

N is a neighborhood of p, so $\exists$ an open set G such that $p \in G \subset N$. By hypothesis, $N \subset M$, so

$$p \in G \subset N \subset M \quad \text{which implies} \quad p \in G \subset M$$

and hence M is a neighborhood of p.

40. Determine whether or not each of the following intervals is a neighborhood of 0 under the usual topology for the real line **R**. (i) $(-\frac{1}{2}, \frac{1}{2}]$, (ii) $(-1, 0]$, (iii) $[0, \frac{1}{2})$, (iv) $(0, 1]$.

Solution:

(i) Note that $0 \in (-\frac{1}{2}, \frac{1}{2}) \subset (-\frac{1}{2}, \frac{1}{2}]$ and $(-\frac{1}{2}, \frac{1}{2})$ is open; so $(-\frac{1}{2}, \frac{1}{2}]$ is a neighborhood of 0.

(ii) and (iii) Any $\mathcal{U}$-open set G containing 0 contains an open interval (a, b) containing 0, i.e. $a < 0 < b$; hence G contains points both greater and less than 0. So neither $(-1, 0]$ nor $[0, \frac{1}{2})$ is a neighborhood of 0.

(iv) The interval $(0, 1]$ does not even contain 0 and hence is not a neighborhood of 0.

41. Prove: A set G is open if and only if it is a neighborhood of each of its points.

Solution:

Suppose G is open; then each point $p \in G$ belongs to the open set G contained in G. Hence G is a neighborhood of each of its points.

Conversely, suppose G is a neighborhood of each of its points. So, for each point $p \in G$, $\exists$ an open set G_p such that $p \in G_p \subset G$. Hence $G = \cup \{G_p : p \in G\}$ and is open since it is a union of open sets.

42. Prove Proposition 5.10: Let $\mathcal{N}_p$ be the neighborhood system of a point p in a topological space X. Then:

(i) $\mathcal{N}_p$ is not empty and p belongs to each member of $\mathcal{N}_p$.

(ii) The intersection of any two members of $\mathcal{N}_p$ belongs to $\mathcal{N}_p$.

(iii) Every superset of a member of $\mathcal{N}_p$ belongs to $\mathcal{N}_p$.

(iv) Each member $N \in \mathcal{N}_p$ is a superset of a member $G \in \mathcal{N}_p$ where G is a neighborhood of each of its points.

Solution:

(i) If $N \in \mathcal{N}_p$, then $\exists$ an open set G such that $p \in G \subset N$; hence $p \in N$. Note $X \in \mathcal{N}_p$ since X is an open set containing p; so $\mathcal{N}_p \neq \emptyset$.

(ii) Proven in Problem 38. (iii) Proven in Problem 39.

(iv) If $N \in \mathcal{N}_p$, then N is a neighborhood of p, so $\exists$ an open set G such that $p \in G \subset N$. But by the preceding problem $G \in \mathcal{N}_p$ and G is a neighborhood of each of its points.

SUBSPACES, RELATIVE TOPOLOGIES

43. Consider the following topology on $X = \{a, b, c, d, e\}$:

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}\}$$

List the members of the relative topology $\mathcal{T}_A$ on $A = \{a, c, e\}$.

Solution:

$\mathcal{T}_A = \{A \cap G : G \in \mathcal{T}\}$, so the members of $\mathcal{T}_A$ are:

$A \cap X = A$ $\quad$ $A \cap \{a\} = \{a\}$ $\quad$ $A \cap \{a,c,d\} = \{a,c\}$ $\quad$ $A \cap \{a,b,e\} = \{a,e\}$

$A \cap \emptyset = \emptyset$ $\quad$ $A \cap \{a,b\} = \{a\}$ $\quad$ $A \cap \{a,b,c,d\} = \{a,c\}$

In other words, $\mathcal{T}_A = \{A, \emptyset, \{a\}, \{a,c\}, \{a,e\}\}$. Observe that $\{a,c\}$ is not open in X, but is relatively open in A, i.e. is $\mathcal{T}_A$-open.

44. Consider the usual topology $\mathcal{U}$ on the real line **R**. Describe the relative topology $\mathcal{U}_N$ on the set **N** of positive integers.

Solution:

Observe that, for each positive integer $n_0 \in \mathbf{N}$,

$$\{n_0\} = \mathbf{N} \cap (n_0 - \tfrac{1}{2}, n_0 + \tfrac{1}{2})$$

and $(n_0 - \frac{1}{2}, n_0 + \frac{1}{2})$ is a $\mathcal{U}$-open set; so every singleton subset $\{n_0\}$ of **N** is open relative to **N**. Hence every subset of **N** is open relative to **N** since it is a union of singleton sets. In other words, $\mathcal{U}_N$ is the discrete topology on **N**.

45. Let A be a $\mathcal{T}$-open subset of $(X, \mathcal{T})$ and let $A \subset Y \subset X$. Show that A is also open relative to the relative topology on Y, i.e. A is a $\mathcal{T}_Y$-open subset of Y.

Solution:

$\mathcal{T}_Y = \{Y \cap G : G \in \mathcal{T}\}$. But $A \subset Y$ and $A \in \mathcal{T}$; so $A = Y \cap A \in \mathcal{T}_Y$.

46. Consider the usual topology $\mathcal{U}$ on the real line **R**. Determine whether or not each of the following subsets of $I = [0,1]$ are open relative to I, i.e. $\mathcal{T}_I$-open: (i) $(\frac{1}{2}, 1]$, (ii) $(\frac{1}{2}, \frac{2}{3})$, (iii) $(0, \frac{1}{2}]$.

Solution:

(i) Note that $(\frac{1}{2}, 1] = I \cap (\frac{1}{2}, 3)$ and $(\frac{1}{2}, 3)$ is open in **R**; hence $(\frac{1}{2}, 1]$ is open relative to I.

(ii) Since $(\frac{1}{2}, \frac{2}{3})$ is open in **R**, i.e. $(\frac{1}{2}, \frac{2}{3}) \in \mathcal{U}$, it is open relative I by the preceding problem. In fact, $(\frac{1}{2}, \frac{2}{3}) = I \cap (\frac{1}{2}, \frac{2}{3})$.

(iii) Since $(0, \frac{1}{2}]$ is not the intersection of I with any $\mathcal{U}$-open subset of **R**, it is not $\mathcal{U}_I$-open.

47. Let A be a subset of a topological space $(X, \mathcal{T})$. Show that the relative topology $\mathcal{T}_A$ is well-defined. In other words, show that $\mathcal{T}_A = \{A \cap G : G \in \mathcal{T}\}$ is a topology on A.

Solution:

Since $\mathcal{T}$ is a topology, X and $\emptyset$ belong to $\mathcal{T}$. Hence $A \cap X = A$ and $A \cap \emptyset = \emptyset$ both belong to $\mathcal{T}_A$, which then satisfies $[\mathbf{O_1}]$.

Now let $\{H_i : i \in I\}$ be a subclass of $\mathcal{T}_A$. By definition of $\mathcal{T}_A$, for each $i \in I$ $\exists$ a $\mathcal{T}$-open set G_i such that $H_i = A \cap G_i$. By the distributive law of intersection over union,

$$\cup_i H_i = \cup_i (A \cap G_i) = A \cap (\cup_i G_i)$$

But $\cup_i G_i \in \mathcal{T}$ as it is the union of $\mathcal{T}$-open sets; hence $\cup_i H_i \in \mathcal{T}_A$. Thus $\mathcal{T}_A$ satisfies $[\mathbf{O_2}]$.

Now suppose $H_1, H_2 \in \mathcal{T}_A$. Then $\exists$ $G_1, G_2 \in \mathcal{T}$ such that $H_1 = A \cap G_1$ and $H_2 = A \cap G_2$. But $G_1 \cap G_2 \in \mathcal{T}$ since $\mathcal{T}$ is a topology. Hence

$$H_1 \cap H_2 = (A \cap G_1) \cap (A \cap G_2) = A \cap (G_1 \cap G_2)$$

belongs to $\mathcal{T}_A$. Accordingly, $\mathcal{T}_A$ satisfies $[\mathbf{O_3}]$ and is a topology on A.

48. Let $(X, \mathcal{T})$ be a subspace of $(Y, \mathcal{T}^*)$ and let $(Y, \mathcal{T}^*)$ be a subspace of $(Z, \mathcal{T}^{**})$. Show that $(X, \mathcal{T})$ is also a subspace of $(Z, \mathcal{T}^{**})$.

Solution:

Since $X \subset Y \subset Z$, $(X, \mathcal{T})$ is a subspace of $(Z, \mathcal{T}^{**})$ if and only if $\mathcal{T}^{**}_X = \mathcal{T}$. Let $G \in \mathcal{T}$; now $\mathcal{T}^*_X = \mathcal{T}$, so $\exists G^* \in \mathcal{T}^*_X$ for which $G = X \cap G^*$. But $\mathcal{T}^* = \mathcal{T}^{**}_Y$, so $\exists G^{**} \in \mathcal{T}^{**}$ such that $G^* = Y \cap G^{**}$. Thus

$$G = X \cap G^* = X \cap Y \cap G^{**} = X \cap G^{**}$$

since $X \subset Y$; so $G \in \mathcal{T}^{**}_X$. Accordingly, $\mathcal{T} \subset \mathcal{T}^{**}_X$.

Now assume $G \in \mathcal{T}^{**}_X$, i.e. $\exists H \in \mathcal{T}^{**}$ such that $G = X \cap H$. But $Y \cap H \in \mathcal{T}^{**}_Y = \mathcal{T}^*$ so $X \cap (Y \cap H) \in \mathcal{T}^*_X = \mathcal{T}$. Since

$$X \cap (Y \cap H) = X \cap H = G$$

we have $G \in \mathcal{T}$. Accordingly, $\mathcal{T}^{**}_X \subset \mathcal{T}$ and the theorem is proved.

MISCELLANEOUS PROBLEMS

49. Let $\mathcal{P}(X)$ be the power set, i.e. class of subsets, of a non-empty set X. Furthermore, let $k : \mathcal{P}(X) \to \mathcal{P}(X)$ be the identity mapping, i.e. for each $A \subset X$, $k(A) = A$.

(i) Verify that k satisfies the Kuratowski Closure Axioms of Theorem 5.12.

(ii) Determine the topology on X induced by k.

Solution:

(i) $k(\emptyset) = \emptyset$, so $[\mathbf{K_1}]$ is satisfied. $k(A \cup B) = A \cup B = k(A) \cup k(B)$, so $[\mathbf{K_3}]$ is satisfied.
$k(A) = A \supset A$, so $[\mathbf{K_2}]$ is satisfied. $k(k(A)) = k(A)$, so $[\mathbf{K_4}]$ is satisfied.

(ii) A subset $F \subset X$ is closed in the topology induced by k if and only if $k(F) = F$. But $k(A) = A$ for every $A \subset X$, so every set is closed and k induces the discrete topology.

50. Let $\mathcal{T}$ be the cofinite topology on the real line $\mathbf{R}$, and let $\langle a_1, a_2, \ldots \rangle$ be a sequence in $\mathbf{R}$ with distinct terms. Show that $\langle a_n \rangle$ converges to every real number $p \in \mathbf{R}$.

Solution:

Let G be any open set containing $p \in \mathbf{R}$. By definition of the cofinite topology, G^c is a finite set and hence can contain only a finite number of the terms of the sequence $\langle a_n \rangle$ since the terms are distinct. Thus G contains almost all of the terms of $\langle a_n \rangle$, and so $\langle a_n \rangle$ converges to p.

51. Let $\mathbf{T}$ be the collection of all topologies on a non-empty set X, partially ordered by class inclusion. Show that $\mathbf{T}$ is a complete lattice, i.e. if $\mathbf{S}$ is a non-empty subcollection of $\mathbf{T}$ then $\sup(\mathbf{S})$ and $\inf(\mathbf{S})$ exist.

Solution:

Let $\mathcal{T}_1 = \cap\{\mathcal{T} : \mathcal{T} \in \mathbf{S}\}$. By Theorem 5.1, $\mathcal{T}_1$ is a topology; so $\mathcal{T}_1 \in \mathbf{T}$ and $\mathcal{T}_1 = \inf(\mathbf{S})$.

Now let $\mathbf{B}$ be the collection of all upper bounds of $\mathbf{S}$. Observe that $\mathbf{B}$ is non-empty since, for example, the discrete topology $\mathcal{D}$ on X belongs to $\mathbf{B}$. Let $\mathcal{T}_2 = \cap\{\mathcal{T} : \mathcal{T} \in \mathbf{B}\}$. Again by Theorem 5.1, $\mathcal{T}_2$ is a topology on X and, furthermore, $\mathcal{T}_2 = \sup(\mathbf{S})$.

52. Let X be a non-empty set and, for each point $p \in X$, let $\mathcal{A}_p$ denote the class of subsets of X containing p.
   (i) Verify that $\mathcal{A}_p$ satisfies the Neighborhood Axioms of Theorem 5.11.
   (ii) Determine the induced topology on X.

Solution:

(i) Since $p \in X$, $X \in \mathcal{A}_p$ and, so, $\mathcal{A}_p \neq \emptyset$. By hypothesis, p belongs to each member of $\mathcal{A}_p$. Hence $[\mathbf{A}_1]$ is satisfied.

If $M, N \in \mathcal{A}_p$, then $p \in M$ and $p \in N$, and so $p \in M \cap N$. Hence $M \cap N \in \mathcal{A}_p$ and so $[\mathbf{A}_2]$ is satisfied.

If $N \in \mathcal{A}_p$ and $N \subset M$, i.e. if $p \in N \subset M$, then $p \in M$. Hence $M \in \mathcal{A}_p$ and so $[\mathbf{A}_3]$ is satisfied.

By definition of $\mathcal{A}_p$, every $A \subset X$ has the property that $A \in \mathcal{A}_p$ for every $p \in A$. Hence $[\mathbf{A}_4]$ is satisfied.

(ii) A subset $A \subset X$ is open in the induced topology if and only if $A \in \mathcal{A}_p$ for every $p \in A$. Since every subset of X has this property, the induced topology on X is the discrete topology.

Supplementary Problems

TOPOLOGICAL SPACES

53. List all possible topologies on the set $X = \{a, b\}$.

54. Prove Theorem 5.1: Let $\{\mathcal{T}_i : i \in I\}$ be any collection of topologies on a set X. Then the intersection $\cap_i \mathcal{T}_i$ is also a topology on X.

55. Let X be an infinite set and let $\mathcal{T}$ be a topology on X in which all infinite subsets of X are open. Show that $\mathcal{T}$ is the discrete topology on X.

56. Let X be an infinite set and let $\mathcal{T}$ consist of $\emptyset$ and all subsets of X whose complements are countable.
   (i) Prove that $(X, \mathcal{T})$ is a topological space.
   (ii) If X is countable, describe the topology determined by $\mathcal{T}$.

57. Let $\mathcal{T} = \{\mathbf{R}^2, \emptyset\} \cup \{G_k : k \in \mathbf{R}\}$ be the class of subsets of the plane $\mathbf{R}^2$ where
$$G_k = \{\langle x, y\rangle : x, y \in \mathbf{R},\ x > y + k\}$$
   (i) Prove that $\mathcal{T}$ is a topology on $\mathbf{R}^2$.
   (ii) Is $\mathcal{T}$ a topology on $\mathbf{R}^2$ if "$k \in \mathbf{R}$" is replaced by "$k \in \mathbf{N}$"? by "$k \in \mathbf{Q}$"?

58. Prove that $(\mathbf{R}^2, \mathcal{T})$ is a topological space where the elements of $\mathcal{T}$ are $\emptyset$ and the complements of finite sets of lines and points.

59. Let $\{p\}$ be an arbitrary singleton set such that $p \notin \mathbf{R}$; e.g. $\{\mathbf{R}\}$. Furthermore, let $\mathbf{R}^* = \mathbf{R} \cup \{p\}$ and let $\mathcal{T}$ be the class of subsets of $\mathbf{R}^*$ consisting of all $\mathcal{U}$-open subsets of $\mathbf{R}$ and the complements (relative to $\mathbf{R}^*$) of all bounded $\mathcal{U}$-closed subsets of $\mathbf{R}$. Prove that $\mathcal{T}$ is a topology on $\mathbf{R}^*$.

60. Let $\{p\}$ be an arbitrary singleton set such that $p \notin \mathbf{R}$; and let $\mathbf{R}^* = \mathbf{R} \cup \{p\}$. Furthermore, let $\mathcal{T}$ be the class of subsets of $\mathbf{R}^*$ consisting of all subsets of $\mathbf{R}$ and the complements (relative to $\mathbf{R}^*$) of all finite subsets of $\mathbf{R}$. Prove that $\mathcal{T}$ is a topology on $\mathbf{R}^*$.

ACCUMULATION POINTS, DERIVED SETS

61. Prove: $A' \cup B' = (A \cup B)'$.

62. Prove: If p is a limit point of the set A, then p is also a limit point of $A \setminus \{p\}$.

63. Prove: Let X be a cofinite topological space. Then A' is closed for any subset A of X.

64. Consider the topological space $(\mathbf{R}, \mathcal{T})$ where $\mathcal{T}$ consists of $\mathbf{R}$, $\emptyset$ and all open infinite intervals $E_a = (a, \infty)$, $a \in \mathbf{R}$. Find the derived set of: (i) the interval $[4, 10)$; (ii) $\mathbf{Z}$, the set of integers.

65. Let $\mathcal{T}$ be the topology on $\mathbf{R}^* = \mathbf{R} \cup \{p\}$ defined in Problem 59.

(i) Determine the accumulation points of the following sets:
(1) open interval (a, b), $a, b \in \mathbf{R}$ (2) infinite open interval (a, ∞), $a \in \mathbf{R}$ (3) $\mathbf{R}$.

(ii) Determine those subsets of $\mathbf{R}^*$ which have p as a limit point.

66. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X with $\mathcal{T}_1$ coarser than $\mathcal{T}_2$, i.e. $\mathcal{T}_1 \subset \mathcal{T}_2$.

(i) Show that every $\mathcal{T}_2$-accumulation point of a subset A of X is also a $\mathcal{T}_1$-accumulation point.

(ii) Construct an example in which the converse of (i) does not hold.

CLOSED SETS, CLOSURE OF A SET, DENSE SUBSETS

67. Construct a non-discrete topological space in which the closed sets are identical to the open sets.

68. Prove: $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. Construct an example in which equality does not hold.

69. Prove: $\bar{A} \setminus \bar{B} \subset \overline{(A \setminus B)}$. Construct an example in which equality does not hold.

70. Prove: If A is open, then $A \cap \bar{B} \subset \overline{A \cap B}$.

71. Prove: Let A be a dense subset of $(X, \mathcal{T})$, and let B be a non-empty open subset of X. Then $A \cap B \neq \emptyset$.

72. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X with $\mathcal{T}_1$ coarser than $\mathcal{T}_2$. Show that the $\mathcal{T}_2$-closure of any subset A of X is contained in the $\mathcal{T}_1$-closure of A.

73. Show that every non-finite subset of an infinite cofinite space X is dense in X.

74. Determine the necessary properties of a space X so that every non-empty open set is dense in X.

INTERIOR, EXTERIOR, BOUNDARY

75. Let X be a discrete space and let $A \subset X$. Find (i) $\text{int}(A)$, (ii) $\text{ext}(A)$, and (iii) $\text{b}(A)$.

76. Prove: (i) $\text{b}(A) \subset A$ if and only if A is closed.
(ii) $\text{b}(A) \cap A = \emptyset$ if and only if A is open.
(iii) $\text{b}(A) = \emptyset$ if and only if A is both open and closed.

77. Prove: If $\bar{A} \cap \bar{B} = \emptyset$, then $\text{b}(A \cup B) = \text{b}(A) \cup \text{b}(B)$.

78. Prove: (i) $A^\circ \cap B^\circ = (A \cap B)^\circ$; (ii) $A^\circ \cup B^\circ \subset (A \cup B)^\circ$. Construct an example in which equality in (ii) does not hold.

79. Prove: $\text{b}(A^\circ) \subset \text{b}(A)$. Construct an example in which equality does not hold.

80. Prove: Let A be any subset of a topological space X. Then $\text{int}(A) \cup \text{ext}(A)$ is dense in X.

81. Prove: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X with $\mathcal{T}_1$ coarser than $\mathcal{T}_2$, i.e. $\mathcal{T}_1 \subset \mathcal{T}_2$, and let $A \subset X$. Then:

(i) The $\mathcal{T}_1$-interior of A is a subset of the $\mathcal{T}_2$-interior of A.

(ii) The $\mathcal{T}_2$-boundary of A is a subset of the $\mathcal{T}_1$-boundary of A.

NEIGHBORHOODS, NEIGHBORHOOD SYSTEMS

82. Let X be a cofinite topological space. Show that every neighborhood of a point $p \in X$ is an open set.

83. Let X be an indiscrete space. Determine the neighborhood system $\mathcal{N}_p$ of any point $p \in X$.

84. Show that if $\mathcal{N}_p$ is finite, then $\bigcap\{N : N \in \mathcal{N}_p\}$ belongs to $\mathcal{N}_p$.

SUBSPACES, RELATIVE TOPOLOGIES

85. Show that every subspace of a discrete space is also discrete.

86. Show that every subspace of an indiscrete space is indiscrete.

87. Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. Show that $E \subset Y$ is $\mathcal{T}_Y$-closed if and only if $E = Y \cap F$, where F is a $\mathcal{T}$-closed subset of X.

88. Let $(A, \mathcal{T}_A)$ be a subspace of $(X, \mathcal{T})$. Prove that $\mathcal{T}_A$ consists of the members of $\mathcal{T}$ contained in A, i.e. $\mathcal{T}_A = \{G : G \subset A, G \in \mathcal{T}\}$, if and only if A is a $\mathcal{T}$-open subset of X.

89. Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. For any subset A of Y, let $\bar{A}$ and A° be the closure and interior of A with respect to $\mathcal{T}$ and let $(\bar{A})_Y$ and $(A^\circ)_Y$ be the closure and interior of A with respect to $\mathcal{T}_Y$. Prove (i) $(\bar{A})_Y = \bar{A} \cap Y$, (ii) $A^\circ = (A^\circ)_Y \cap Y^\circ$.

90. Let A, B and C be subsets of a topological space X with $C \subset A \cup B$. If A, B and $A \cup B$ are given the relative topologies, prove that C is open with respect to $A \cup B$ if and only if $C \cap A$ is open with respect to A and $C \cap B$ is open with respect to B.

EQUIVALENT DEFINITIONS OF TOPOLOGIES

91. Prove Theorem 5.11: Let X be a non-empty set and let there be assigned to each point $p \in X$ a class $\mathcal{A}_p$ of subsets of X satisfying the following axioms:

[A$_1$] $\mathcal{A}_p$ is not empty and p belongs to each member of $\mathcal{A}_p$.
[A$_2$] The intersection of any two members of $\mathcal{A}_p$ belongs to $\mathcal{A}_p$.
[A$_3$] Every superset of a member of $\mathcal{A}_p$ belongs to $\mathcal{A}_p$.
[A$_4$] Each member $N \in \mathcal{A}_p$ is a superset of a member $G \in \mathcal{A}_p$ such that $G \in \mathcal{A}_g$ for every $g \in G$.

Then there exists one and only one topology $\mathcal{T}$ on X such that $\mathcal{A}_p$ is the $\mathcal{T}$-neighborhood system of the point $p \in X$.

92. Prove Theorem 5.12: Let X be a non-empty set and let $k : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfy the following Kuratowski Closure Axioms:

$$[\mathbf{K_1}]\ k(\emptyset) = \emptyset, \quad [\mathbf{K_2}]\ A \subset k(A), \quad [\mathbf{K_3}]\ k(A \cup B) = k(A) \cup k(B), \quad [\mathbf{K_4}]\ k(k(A)) = k(A)$$

Then there exists one and only one topology $\mathcal{T}$ on X such that $k(A)$ will be the $\mathcal{T}$-closure of $A \subset X$.

93. Prove: Let X be a non-empty set and let $i : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfy the following properties:

$$\text{(i) } i(X) = X, \quad \text{(ii) } i(A) \subset A, \quad \text{(iii) } i(A \cup B) = i(A) \cup i(B), \quad \text{(iv) } i(i(A)) = i(A)$$

Then there exists one and only one topology $\mathcal{T}$ on X such that $i(A)$ will be the $\mathcal{T}$-interior of $A \subset X$.

94. Prove: Let X be a non-empty set and let $\mathcal{F}$ be a class of subsets of X satisfying the following properties:

(i) X and $\emptyset$ belong to $\mathcal{F}$.
(ii) The intersection of any number of members of $\mathcal{F}$ belongs to $\mathcal{F}$.
(iii) The union of any two members of $\mathcal{F}$ belongs to $\mathcal{F}$.

Then there exists one and only one topology $\mathcal{T}$ on X such that the members of $\mathcal{F}$ are precisely the $\mathcal{T}$-closed subsets of X.

95. Let a neighborhood of a real number $p \in \mathbf{R}$ be any set containing p and containing all the rational numbers of some open interval (a, b) where $a < p < b$.

(i) Show that these neighborhoods actually satisfy the neighborhood axioms and hence define a topology on the real line $\mathbf{R}$.
(ii) Show that any set of irrational numbers does not contain any accumulation points.
(iii) Show that any sequence of irrational numbers, such as $\langle \pi/2, \pi/3, \pi/4, \ldots \rangle$, does not converge.

Answers to Supplementary Problems

53. $\{X, \emptyset\}$, $\{X, \{a\}, \emptyset\}$, $\{X, \{b\}, \emptyset\}$ and $\{X, \{a\}, \{b\}, \emptyset\}$.

56. (ii) Discrete topology.

64. (i) $(-\infty, 10]$ (ii) $\mathbf{R}$

65. (i): (1) $[a, b]$, (2) $[a, \infty) \cup \{p\}$, (3) $\mathbf{R}^*$. (ii) Unbounded subsets of $\mathbf{R}$.

67. $X = \{a, b, c\}$, $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{c\}\}$

74. X is an indiscrete space.

75. (i) A, (ii) A^c, (iii) $\emptyset$